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# LARGE CLASSES WITH THE FIXED POINT PROPERTY IN A DEGENERATE LORENTZ-MARCINKIEWICZ SPACE

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Abstract. Recently, Nezir has renormed $\ell^1$ and observed that the resulting space turns out be a degenerate Lorentz-Marcinkiewicz space. Then, fixed point properties have been investigated for the space, its dual and its predual. Also, inspiring from the study of Goebel and Kuczumow, as they showed for the Banach space of absolutely summable sequences $\ell^1$, Nezir showed that a class of non-weak* compact, closed, convex and bounded sets in one of these spaces has the fixed point property for affine nonexpansive mappings. In fact, very recently, generalizing the equivalent norm on $\ell^1$, Nezir and Mustafa obtained new type of degenerate Lorentz-Marcinkiewicz spaces with their fixed point properties and got the analogy of Goebel and Kuczumow's for the resulting space. In this paper, we show that there exists large classes of non-weak* compact, closed, convex and bounded sets with the fixed point property for affine nonexpansive mappings in the generalized degenerate Lorentz-Marcinkiewicz space.

Keywords: nonexpansive mapping, nonreflexive Banach space, fixed point property, closed bounded convex set, Lorentz-Marcinkiewicz spaces.

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## 1. Introduction

There is strong relationship between the concept of reflexive Banach spaces and Banach spaces having the fixed point property. It has been questioned whether the two concepts are equivalent unconditionally or depending on some conditions.

It has been observed that most non-reflexive classical Banach spaces such as $\ell^1$ and $c_0$ fail the fixed point property and it is still and has been an open question for over 50 years whether or not all nonreflexive Banach spaces can be renormed to have the fixed point property. It was proved by Lin's result [8] that non-reflexive Banach space $\ell^1$, Banach space of absolutely summable sequences can be renormed to have the fixed point property for nonexpansive mappings.

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Researchers are interested to questionize whether or not there exist other non-reflexive Banach spaces than $\ell^1$ which can be renormed to have the fixed point property for nonexpansive mappins. But then it is necessary to recall that Lin inspired by the work of Goebel and Kuczumow [6] that can be said to let him succeed in his research by using some of strategies of theirs. By Goebel and Kuczumow's study, existence of a large class of non-weak* closed, bounded and convex subsets with fixed point property in $\ell^1$ was showed and later in 2003, it was showed by Kaczor and Prus [7] that under an extra assumption, the sets developed by Goebel and Kuczumow has the fixed point property for asymptotically nonexpansive mappings.

In [12], Nezir recently constructed an equivalent renorming of $\ell^1$ which turns out to produce a degenerate $\ell^1$-analog Lorentz-Marcinkiewicz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}} = (2,1,1,1,\cdots)$ is a decreasing positive sequence in $\ell^\infty\backslash c_0$, rather than in $c_0\backslash\ell^1$ (the usual Lorentz situation). Then, he obtained its isometrically isomorphic predual $\ell^0_{\delta,\infty}$ and dual $\ell_{\delta,\infty}$, corresponding degenerate $c_0$-analog and $\ell^\infty$-analog Lorentz-Marcinkiewicz spaces, respectively. Then, he investigated all types of fixed point properties such as weak, weak*, and regular fixed point property.

Then, very recently, generalizing Nezir's work by constructing another equivalent norm on $\ell^1$ and obtaining his generalized degenerate $\ell^1$-analog Lorentz-Marcinkiewicz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}} = (\alpha+\beta,\beta,\beta,\beta,\cdots),\ for\ \beta\geq\alpha>0$, in [14], Nezir and Mustafa has showed that $\ell_{\delta,1}$ fails to have the fixed point property for nonexpansive mappings but also they show that there exists an infinite dimensional subspace of $\ell_{\delta,1}$ with the fixed point property for affine nonexpansive mappings.

We need to note that Nezir [13] recently also worked on Lorentz sequence spaces and obtained the result that there exists an infinite dimensional subspace of $\ell^1$ analogue Lorentz sequence space with the fixed point property for affine nonexpansive mappings. We believe that similar result can be obtained for the regular Lorentz-Marcinkiewicz space as well. Furthermore, we remark that most of this work forms part of the master thesis of M. Yazici [4].

In this study, we show that it is possible to obtain different large classes with the fixed point property for affine nonexpansive mappings in these spaces. Readers may find it useful to see [2, 3] for various other summable spaces and their properties to extend similar results to such classes of spaces.

## 2. Preliminaries

**Definition 2.1.** *Let $(X,\|\cdot\|)$ be a Banach space and $C$ be a non-empty closed, bounded, convex subset.*

1. *If $T: C\longrightarrow C$ is a mapping such that for all $\lambda\in[0,1]$ and for all $x,y\in C$, $T\big((1-\lambda)\,x+\lambda\,y\big) = (1-\lambda)\,T(x)+\lambda\,T(y)$ . then $T$ is said to be an affine mapping.*
2. *If $T: C\longrightarrow C$ is a mapping such that $\|T(x)-T(y)\|\leq\|x-y\|$ , $for\ all\ x,y\in C$ then $T$ is said to be a nonexpansive mapping.*

   *Also, if for every nonexpansive mapping $T: C\longrightarrow C$, there exists $x\in C$ with $T(x)=x$, then $X$ is said to have the fixed point property for nonexpansive mappings.*

We should note that in the V. Nezir's Ph.D. thesis [11], written under supervision of C. Lennard, the usual Lorentz-Marcienkiewicz spaces and their fixed point properties were studied; hence, we can give their definitions below to understand how different the degenerate ones are.

Let $w \in (c_0\backslash\ell^1)^+$, $w_1 = 1$ and $(w_n)_{n\in\mathbb{N}}$ be decreasing; that is, consider a scalar sequence given by $w = (w_n)_{n\in\mathbb{N}}$, $w_n > 0, \forall n \in \mathbb{N}$ such that $1 = w_1 \geq w_2 \geq w_3 \geq \cdots \geq w_n \geq w_{n+1} \geq \ldots, \forall n \in \mathbb{N}$ with $w_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sum_{n=1}^{\infty} w_n = \infty$. This sequence is called a weight sequence. For example, $w_n = \frac{1}{n}, \forall n \in \mathbb{N}$.

**Definition 2.2.** $l_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|x\|_{w,\infty} := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} {x_j}^\star}{\sum_{j=1}^{n} w_j} < \infty \right\}.$

*Here, $x^\star$ represents the decreasing rearrangement of the sequence $x$, which is the sequence of $|x| = (|x_j|)_{j\in\mathbb{N}}$, arranged in non-increasing order, followed by infinitely many zeros when $|x|$ has only finitely many non-zero terms. This space is non-separable and an analogue of $l_\infty$ space.*

**Definition 2.3.** $l^0_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \limsup_{n\longrightarrow\infty} \frac{\sum_{j=1}^{n} {x_j}^\star}{\sum_{j=1}^{n} w_j} = 0 \right\}.$

*This is a separable subspace of $l_{w,\infty}$ and an analogue of $c_0$ space.*

**Definition 2.4.** $l_{w,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|x\|_{w,1} := \sum_{j=1}^{\infty} w_j\, {x_j}^\star < \infty \right\}.$

*This is a separable subspace of $l_{w,\infty}$ and an analogue of $l_1$ space with the following facts: $(l^0_{w,\infty})^\star \cong l_{w,1}$ and $(l_{w,1})^\star \cong l_{w,\infty}$ where the star denotes the dual of a space while $\cong$ denotes isometrically isomorphic.*

More information about Lorentz spaces can be seen in [10, 9].

Now, we will introduce Nezir's construction. For all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, we define $|||x||| := \|x\|_1 + \|x\|_\infty = \sum_{n=1}^{\infty} |x_n| + \sup_{n\in\mathbb{N}} |x_n|$ . Clearly $|||\cdot|||$ is an equivalent norm on $\ell^1$ with $\|x\|_1 \leq |||x||| \leq 2\|x\|_1$, $\forall x \in \ell^1$. Note that $\forall x \in \ell^1$, $|||x||| = 2x_1^* + x_2^* + x_3^* + x_4^* + \cdots$ where $z^*$ is the decreasing rearrangement of $|z| = (|z_n|)_{n\in\mathbb{N}}$, $\forall z \in c_0$. Let $\delta_1 := 2, \delta_2 := 1, \delta_3 := 1, \cdots, \delta_n := 1$, $\forall n \geq 4$. We see that $(\ell^1, |||\cdot|||)$ is a (degenerate) Lorentz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}}$ is a decreasing positive sequence in $\ell^\infty\backslash c_0$, rather than in $c_0\backslash\ell^1$ (the usual Lorentz situation).

The following lemma [12] will be main ingredient in our theorems.

**Lemma 2.1.** *Let $(X, \|.\|)$ be a Banach space.*

1. *If $X$ has the Banach–Saks property and $x \in X$ is the weak limit of a bounded sequence $(x_n)_n$, then there exists a subsequence $(x_{n_k})_k$ whose Cesaro norm limit is $x$ such that if $s$ is defined by* $s(y) = \limsup_m \left\| \frac{1}{m} \sum_{k=1}^{m} x_{n_k} - y \right\|$ , $\forall y \in X$, *then we have* $s(x) = 0$ *and* $s(y) = \|y - x\|$ , $\forall y \in X$.
2. *If $X$ has the weak Banach–Saks property and $x \in X$ is the weak limit of the sequence $(x_n)_n$, then there exists a subsequence $(x_{n_k})_k$ whose Cesaro norm limit is $x$ such that if $s$ is defined by* $s(y) = \limsup_m \left\| \frac{1}{m} \sum_{k=1}^{m} x_{n_k} - y \right\|$ , $\forall y \in X$, *then we have* $s(x) = 0$ *and* $s(y) = \|y - x\|$ , $\forall y \in X$.

*Hence, due to the weak Banach-Saks property of our space, which can be deduced by the works* [16, 17, 1], *the above applies.*

## 3. Main Result

Generalizing Nezir's construction, Nezir and Mustafa recently constructed another equivalent norm by their conference paper submitted and entitled "On the fixed point property for a degenerate Lorentz-Marcinkiewicz Space" as the following: let $\beta \geq \alpha > 0$. For all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, they define $|||x||| := \beta\|x\|_1 + \alpha\|x\|_\infty = \beta \sum\limits_{n=1}^{\infty} |x_n| + \alpha \sup\limits_{n\in\mathbb{N}} |x_n|$ . Clearly $||| \cdot |||$ is an equivalent norm on $\ell^1$ with $\beta\|x\|_1 \leq |||x||| \leq (\alpha+\beta)\|x\|_1$, $\forall x \in \ell^1$. Note that $\forall x \in \ell^1$, $|||x||| = \beta\left(\frac{\alpha+\beta}{\beta}x_1^* + x_2^* + x_3^* + x_4^* + \cdots\right)$ where $z^*$ is the decreasing rearrangement of $|z| = (|z_n|)_{n\in\mathbb{N}}$, $\forall z \in c_0$.

Let $\delta_1 := (\alpha+\beta), \delta_2 := \beta, \delta_3 := \beta, \cdots, \delta_n := \beta$, $\forall n \geq 4$. Then, they see that $(\ell^1, ||| \cdot |||)$ is a (degenerate) Lorentz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}}$ is a decreasing positive sequence in $\ell^\infty \backslash c_0$, rather than in $c_0 \backslash \ell^1$ (the usual Lorentz situation).

Using their construction, we show that there exists large classes of nonweakly* compact, closed, bounded and convex subsets of $\ell_{\delta,1} = (\ell^1, ||| \cdot |||)$ with the fixed point property for nonexpansive mappings [fpp(n.e.)] using the ideas of Goebel and Kuczumow [6] where they show that there exists a large class of nonweakly* compact, closed, bounded and convex subsets of $(\ell^1, \|\cdot\|_1$ with fpp(n.e.). Note that the following example and the first theorem below that were given by Nezir and Mustafa. Our job is to come with new examples with theorems and more examples can be given in future projects.

**Example 3.1.** *Fix $b \in (0,1)$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in $c_0$ by setting $f_1 := b\,e_1$, $f_2 := b\,e_2$ and $f_n := e_n$, $\forall n \geq 3$. Next, define the closed, bounded, convex subset $E = E_b$ of $\ell^1$ by $E := \left\{\sum_{n=1}^{\infty} t_n f_n : \text{ each } t_n \geq 0 \text{ and } \sum\limits_{n=1}^{\infty} t_n = 1\right\}$ .*

**Theorem 3.1.** *The set $E$ defined as in the example above has the fixed point property for $|||.|||$-nonexpansive mappings where the norm $|||\cdot|||$ on $\ell^1$ is given as follows: $|||x||| = \beta\|x\|_1 + \alpha\|x\|_\infty$, for $\beta \geq \alpha > 0$ and $\forall x \in \ell^1$.*

**Example 3.2.** *Let $b_1,\ b_2 \in (0,1)$, $2b_1 \geq b_2$ and $b_2 \geq b_1$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in $c_0$ by setting $f_1 := b_1\,e_1$, $f_2 := b_2\,e_2$ and $f_n := e_n$, for all integers $n \geq 3$. Next, define the closed, bounded, convex subset $E$ of $\ell^1$ as above.*

**Theorem 3.2.** *The set $E$ defined as in the example above has the fixed point property for affine $|||.|||$-nonexpansive mappings.*

*Proof.* We will consider $b_2 > b_1$ firstly. Proof of the case of the equality is obtained by the previous theorem but even imitating the proof below, one would get the result for the equality case. We will be using the proof steps of the last theorem in [12] derived from Goebel and Kuczumow's study [6] given in detail as in Everest's PhD thesis [5], written under the supervision of Lennard. Let $T : E \to E$ be a nonexpansive mapping. Then there exists a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}} \in E$ such that $\left|\left|\left|Tx^{(n)} - x^{(n)}\right|\right|\right| \underset{n}{\to} 0$ and so $\left\|Tx^{(n)} - x^{(n)}\right\|_1 \underset{n}{\to} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $z \in \ell^1$ such that $x^{(n)}$ converges to $z$ in weak* topology. Then, by Lemma 2.1, we can define a function $s : \ell^1 \longrightarrow [0,\infty)$ by $s(y) = \limsup\limits_m \left|\left|\left|\frac{1}{m}\sum\limits_{k=1}^{m} x^{(k)} - y\right|\right|\right|$ , $\forall y \in \ell^1$ and so $s(y) = |||y - z|||$ , $\forall y \in \ell^1$. Next, define $W := \overline{E}^{w^*} = \left\{\sum\limits_{n=1}^{\infty} t_n f_n : \text{ each } t_n \geq 0 \text{ and } \sum\limits_{n=1}^{\infty} t_n \leq 1\right\}$.

*Case 1: $z \in E$.*

Then, we have $s(Tz) = |||Tz - z|||$, and since $T$ is affine and $|||.|||$-nonexpansive.

$$\begin{aligned} s(Tz) &\leq \limsup_m \left|\left|\left| Tz - T\left(\frac{1}{m}\sum_{k=1}^m x^{(k)}\right)\right|\right|\right| + \limsup_m \left|\left|\left| \frac{1}{m}\sum_{k=1}^m x^{(k)} - \frac{1}{m}\sum_{k=1}^m Tx^{(k)}\right|\right|\right| \\ &\leq \limsup_m \left|\left|\left| z - \frac{1}{m}\sum_{k=1}^m x^{(k)}\right|\right|\right| \\ &= s(z). \end{aligned}$$

Therefore, $|||z - Tz||| \leq 0$ and so $Tz = z$.
*Case 2:* $z \in W \setminus E$.

Then $z$ is of the form $\sum\limits_{n=1}^{\infty} \gamma_n f_n$ such that $\sum\limits_{n=1}^{\infty} \gamma_n \ < 1$ and $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. Define $\delta := 1 - \sum_{n=1}^{\infty} \gamma_n$ and next define $h_\lambda := (\gamma_1 + \lambda\delta) f_1 + (\gamma_2 + (1-\lambda)\delta) f_2 + \sum_{n=3}^{\infty} \gamma_n f_n$. We want $h_\lambda$ to be in $E$, so we restrict values of $\lambda$ to be in $\left[-\frac{\gamma_1}{\delta},\ \frac{\gamma_2}{\delta}+1\right]$, and then

$$\begin{aligned} |||h_\lambda - z||| &= \alpha\delta \max\{|\lambda| b_1, |1-\lambda| b_2\} + \beta b_1 \delta |\lambda| + \beta b_2 \delta |1-\lambda| \\ &= \max \begin{cases} (\alpha+\beta) b_2 \delta (1-\lambda) - \beta b_1 \delta \lambda & if\ \lambda \in \left[-\frac{\gamma_1}{\delta},\ 0\right), \\ (\alpha+\beta) b_2 \delta (1-\lambda) + \beta b_1 \delta \lambda & if\ \lambda \in \left[0,\ \frac{b_2}{b_1+b_2}\right), \\ (\alpha+\beta) b_1 \lambda \delta + \beta (1-\lambda) b_2 \delta & if\ \lambda \in \left[\frac{b_2}{b_1+b_2},\ 1\right), \\ (\alpha+\beta) b_1 \lambda \delta + \beta b_2 (\lambda - 1)\delta & if\ \lambda \in \left[1,\ \frac{1}{b_2-b_1}\right)\ and\ if\ \frac{1}{b_2-b_1} \leq \frac{\gamma_2}{\delta}+1, \\ (\alpha+\beta) b_2 (\lambda-1)\delta + \beta b_1 \lambda \delta & if\ \lambda \in \left[\frac{1}{b_2-b_1},\ \frac{\gamma_2}{\delta}+1\right]\ and\ if\ \frac{1}{b_2-b_1} \leq \frac{\gamma_2}{\delta}+1, \\ (\alpha+\beta) b_1 \lambda \delta + \beta b_2 (\lambda-1)\delta & if\ \lambda \in \left[1,\ \frac{\gamma_2}{\delta}+1\right]\ and\ if\ \frac{1}{b_2-b_1} > \frac{\gamma_2}{\delta}+1\ . \end{cases} \end{aligned}$$

Define $\Gamma := \min\limits_{\lambda \in \left[-\frac{\gamma_1}{\delta},\ \frac{\gamma_2}{\delta}+1\right]} |||h_\lambda - z|||$. Therefore, $|||h_\lambda - z|||$ is minimized when $\lambda \in [0,1]$ with unique minimizer such that its minimum value would be $\Gamma = \frac{(\alpha+2\beta) b_1 b_2 \delta}{b_1+b_2}$. Now fix $y \in E$ of the form $\sum\limits_{n=1}^{\infty} t_n f_n$ such that $\sum\limits_{n=1}^{\infty} t_n \ = 1$ with $t_n \geq 0$, $\forall n \in \mathbb{N}$.
*Subcase 2.1:* $b_1|t_1 - \gamma_1| \geq b_2|t_2 - \gamma_2|$ and $b_1|t_1 - \gamma_1| \geq |t_k - \gamma_k|$, $\forall k \geq 3$.
Then,

$$\begin{aligned} |||y - z||| &= ((\alpha+\beta) b_1 - \beta b_2)|t_1 - \gamma_1| + \beta b_2 \sum_{k=1}^{\infty} |t_k - \gamma_k| + \beta(1-b_2)\sum_{k=3}^{\infty} |t_k - \gamma_k| \\ &\geq ((\alpha+\beta) b_1 - \beta b_2)|t_1 - \gamma_1| + \beta b_2 \delta + (1-b_2)|\delta - (t_1 - \gamma_1) - (t_2 - \gamma_2)|. \end{aligned}$$

*Subcase 2.1.1:* Assume $\frac{b_2\delta}{b_1+b_2} \geq |t_1 - \gamma_1|$.
Then clearly the last inequality from above says that

$$\begin{aligned} |||y - z||| &\geq ((\alpha+\beta) b_1 - \beta b_2)|t_1 - \gamma_1| + \beta\delta + \beta(1-b_2)(-1 - \frac{b_1}{b_2})|t_1 - \gamma_1| \\ &\geq \left[\left(((\alpha+\beta) b_1 - \beta b_2) - \beta(1-b_2)\left(1 + \frac{b_1}{b_2}\right)\right)\frac{b_2}{b_1+b_2} + \beta\right]\delta \\ &\geq \frac{(\alpha+2\beta) b_1 b_2 \delta}{b_1+b_2}. \end{aligned}$$

*Subcase 2.1.2:* Assume $\frac{b_2\delta}{b_1+b_2} < |t_1 - \gamma_1|$.

Then,

$$\|\!|y-z|\!\| \geq ((\alpha+\beta)b_1-\beta b_2)|t_1-\gamma_1|+\beta b_2\delta+\beta(1-b_2)\sum_{k=3}^{\infty}|t_k-\gamma_k| \geq \frac{(\alpha+2\beta)b_1b_2\delta}{b_1+b_2}.$$

*Subcase 2.2:* $b_2|t_2-\gamma_2| \geq b_1|t_1-\gamma_1|$ and $b_2|t_2-\gamma_2| \geq |t_k-\gamma_k|$, $\forall k \geq 3$.
Then,

$$\begin{aligned}\|\!|y-z|\!\| &= \alpha b_2|t_2-\gamma_2|+\beta(b_1-b_2)|t_1-\gamma_1|+\beta b_2|t_1-\gamma_1|+\beta b_2|t_2-\gamma_2|+\beta\sum_{k=3}^{\infty}|t_k-\gamma_k| \\ &\geq [(\alpha+\beta)b_1-\beta b_2]\frac{b_2}{b_1}|t_2-\gamma_2|+\beta b_2\sum_{k=1}^{\infty}|t_k-\gamma_k|+\beta(1-b_2)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\ &\geq [(\alpha+\beta)b_1-\beta b_2]|t_2-\gamma_2|+\beta b_2\delta+\beta(1-b_2)|\delta-(t_1-\gamma_1)-(t_2-\gamma_2)|.\end{aligned}$$

*Subcase 2.2.1:* Assume $\frac{b_2\delta}{b_1+b_2} \geq |t_2-\gamma_2|$.
Then clearly the last inequality from above says that

$$\begin{aligned}\|\!|y-z|\!\| &\geq [(\alpha+\beta)b_1-\beta b_2]|t_2-\gamma_2|+\beta\delta+\beta(1-b_2)(-1-\frac{b_2}{b_1})|t_2-\gamma_2| \\ &\geq \left((\alpha+\beta)b_1-\beta b_2-\beta(1-b_2)\left(1+\frac{b_2}{b_1}\right)\right)|t_2-\gamma_2|+\beta\delta.\end{aligned}$$

If $\left((\alpha+\beta)b_1-\beta b_2-\beta(1-b_2)\left(1+\frac{b_2}{b_1}\right)\right) \geq 0$, then, since $b_2 < \frac{2}{3}$ and $\alpha \leq \beta$, we have that $\|\!|y-z|\!\| \geq \frac{(\alpha+2\beta)b_1b_2\delta}{b_1+b_2}$. If $\left((\alpha+\beta)b_1-\beta b_2-\beta(1-b_2)\left(1+\frac{b_2}{b_1}\right)\right) < 0$, then due to $\frac{b_2\delta}{b_1+b_2} \geq |t_2-\gamma_2|$ we get

$$\begin{aligned}\|\!|y-z|\!\| &\geq \left[\left((\alpha+\beta)b_1-\beta b_2-\beta(1-b_2)\left(1+\frac{b_2}{b_1}\right)\right)\frac{b_2\delta}{b_1+b_2}+\beta\right]\delta \\ &\geq \frac{(\alpha+2\beta)b_1b_2\delta}{b_1+b_2}.\end{aligned}$$

*Subcase 2.2.2:* Assume $\frac{b_2\delta}{b_1+b_2} < |t_2-\gamma_2|$.
Then,

$$\begin{aligned}\|\!|y-z|\!\| &\geq [(\alpha+\beta)b_1-\beta b_2]|t_2-\gamma_2|+\beta b_2\delta+\beta(1-b_2)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\ &\geq \frac{(\alpha+2\beta)b_1b_2\delta}{b_1+b_2}.\end{aligned}$$

*Subcase 2.3:* $|t_3-\gamma_3| \geq b_1|t_1-\gamma_1|$, $|t_3-\gamma_3| \geq b_2|t_2-\gamma_2|$, and $|t_3-\gamma_3| \geq |t_k-\gamma_k|$, $\forall k \geq 4$.
Then,

$$\begin{aligned}\|\!|y-z|\!\| &\geq \alpha|t_3-\gamma_3|+\beta(b_1-b_2)\frac{1}{b_1}|t_3-\gamma_3|+\beta b_2|t_1-\gamma_1|+\beta b_2|t_2-\gamma_2|+\beta\sum_{k=3}^{\infty}|t_k-\gamma_k| \\ &\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_3-\gamma_3|+\beta b_2|t_1-\gamma_1|+\beta b_2|t_2-\gamma_2|+\beta\sum_{k=3}^{\infty}|t_k-\gamma_k| \\ &\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_3-\gamma_3|+\beta b_2\delta+\beta(1-b_2)\sum_{k=3}^{\infty}|t_k-\gamma_k|.\end{aligned}$$

Thus,

$$\begin{aligned}\|\!|y-z|\!\|&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_3-\gamma_3|+\beta b_2\delta+\beta(1-b_2)\delta-\beta(1-b_2)\frac{1}{b_1}|t_3-\gamma_3|-\beta(1-b_2)\frac{1}{b_2}|t_3-\gamma_3|\\&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_3-\gamma_3|+\beta b_2\delta+\beta(1-b_2)\delta-\beta(1-b_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)|t_3-\gamma_3|.\end{aligned}$$

*Subcase 2.3.1:* Assume $\frac{b_1b_2\delta}{b_1+b_2}\geq|t_3-\gamma_3|$.

Then clearly the last inequality from above says that

$$\begin{aligned}\|\!|y-z|\!\|&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_3-\gamma_3|+\beta b_2\delta+\beta(1-b_2)\delta-\beta(1-b_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)|t_3-\gamma_3|\\&\geq \left[\frac{(\alpha+\beta)b_1-\beta b_2}{b_1}-\beta(1-b_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)\right]|t_3-\gamma_3|+\beta\delta\\&\geq \left[\frac{(\alpha+\beta)b_1-\beta b_2}{b_1}-\beta(1-b_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)+1\right]\delta\\&\geq \frac{(\alpha+2\beta)b_1b_2\delta}{b_1+b_2}.\end{aligned}$$

*Subcase 2.3.2:* Assume $\frac{b_1b_2\delta}{b_1+b_2}<|t_3-\gamma_3|$.

Then

$$\begin{aligned}\|\!|y-z|\!\|&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_3-\gamma_3|+\beta b_2\delta+\beta(1-b_2)\sum_{k=3}^{\infty}|t_k-\gamma_k|\\&\geq \frac{(\alpha+2\beta)b_1b_2\delta}{b_1+b_2}.\end{aligned}$$

*Subcase 2.4:* $|t_4-\gamma_4|\geq b_1|t_1-\gamma_1|$, $|t_4-\gamma_4|\geq b_2|t_2-\gamma_2|$, and $|t_4-\gamma_4|\geq|t_k-\gamma_k|$, $\forall k\geq 5$ and for $k=3$.

Then,

$$\begin{aligned}\|\!|y-z|\!\|&\geq \alpha|t_4-\gamma_4|+\beta(b_1-b_2)\frac{1}{b_1}|t_4-\gamma_4|+\beta b_2|t_1-\gamma_1|+\beta b_2|t_2-\gamma_2|+\beta\sum_{k=3}^{\infty}|t_k-\gamma_k|\\&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_4-\gamma_4|+\beta b_2|t_1-\gamma_1|+\beta b_2|t_2-\gamma_2|+\beta\sum_{k=3}^{\infty}|t_k-\gamma_k|\\&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_4-\gamma_4|+\beta b_2\delta+\beta(1-b_2)\sum_{k=3}^{\infty}|t_k-\gamma_k|\\&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_4-\gamma_4|+\beta b_2\delta+\beta(1-b_2)|\delta-(t_1-\gamma_1)-(t_2-\gamma_2)|\\&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_4-\gamma_4|+\beta b_2\delta+\beta(1-b_2)\delta-\beta(1-b_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)|t_4-\gamma_4|.\end{aligned}$$

*Subcase 2.4.1:* Assume $\frac{b_1b_2\delta}{b_1+b_2}\geq|t_4-\gamma_4|$.

Then clearly the last inequality from above says that

$$\begin{aligned}\|\!|y-z|\!\|&\geq \frac{(\alpha+\beta)b_1-\beta b_2}{b_1}|t_4-\gamma_4|+\beta b_2\delta+\beta(1-b_2)\delta-\beta(1-b_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)|t_4-\gamma_4|\\&\geq \left[\frac{(\alpha+\beta)b_1-\beta b_2}{b_1}-\beta(1-b_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)\right]|t_4-\gamma_4|+\beta\delta\\&\geq \left[\frac{(\alpha+\beta)b_1-\beta b_2}{b_1}-\beta(1-b_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)+\beta\right]\delta\\&\geq \frac{(\alpha+2\beta)b_1b_2\delta}{b_1+b_2}.\end{aligned}$$

*Subcase 2.4.2:* Assume $\frac{b_1 b_2 \delta}{b_1+b_2} < |t_4 - \gamma_4|$.

Then

$$\begin{aligned} |||y-z||| &\geq \frac{(\alpha+\beta)b_1 - \beta b_2}{b_1}|t_4-\gamma_4| + \beta b_2 \delta + \beta(1-b_2)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\ &\geq \frac{(\alpha+2\beta)b_1 b_2 \delta}{b_1+b_2}. \end{aligned}$$

Therefore, when $\lambda$ is choosen to be in $[0,1]$, for any $y \in E$ and for $z \in W \setminus E$, $|||y-z||| \geq \Gamma$ such that there exists unique $\lambda_0 \in [0,1]$ with $|||h_{\lambda_0} - z|||=\Gamma$.

Now define a subset in our set by $\Lambda := \{y : |||y-z||| \leq \Gamma\}$. Note that $\Lambda \subseteq E$ is a nonempty compact convex subset such that for any $h \in \Lambda$, since $T$ is affine and $|||.|||$-nonexpansive,

$$\begin{aligned} s(Th) &\leq \limsup_m \left|\left|\left| Th - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right|\right|\right| + \limsup_m \left|\left|\left| \frac{1}{m}\sum_{k=1}^{m} x^{(k)} - \frac{1}{m}\sum_{k=1}^{m} Tx^{(k)} \right|\right|\right| \\ &\leq \limsup_m \left|\left|\left| h - \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \right|\right|\right| \\ &= s(h). \end{aligned}$$

Also, $s(Th) = |||z - Th|||$ and $s(h) = |||z-h|||$. Hence,

$$\begin{aligned} |||z-Th||| \leq |||z-h||| &\implies |||z-Th||| = |||z-h||| \\ &\implies Th \in \Lambda. \end{aligned}$$

Therefore, $T(\Lambda) \subseteq \Lambda$, and since $T$ is continuous, Schauder's fixed point theorem [18] tells us that $T$ has a fixed point such that $h = h_{\lambda_0}$ is the unique minimizer of $|||y-z||| \ : \ y \in E$ and $Th = h$.

Therefore, $E$ has fpp(ne) as desired. $\square$

Now, we work on different examples.

**Example 3.3.** *Fix $b \in \left(0, \frac{2}{3}\right)$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in $c_0$ by setting $f_1 := b\, e_1$, $f_2 := b\, e_2$, $f_3 := b\, e_3$ and $f_n := e_n$, for all integers $n \geq 4$. Next, define the closed, bounded, convex subset $E = E_b$ of $\ell^1$ by $E := \left\{ \sum_{n=1}^{\infty} t_n f_n : \text{ each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\}$.*

**Theorem 3.3.** *The set $E$, defined in the example above, has the fixed point property for affine $|||.|||$-nonexpansive mappings.*

*Proof.* We use similar strategy to the one in the proof of Theorem 3.2 and we only have different Case 2 as follows:

*Case 2: $z \in W \setminus E$.*

Then, $z$ is of the form $\sum_{n=1}^{\infty} \gamma_n f_n$ such that $\sum_{n=1}^{\infty} \gamma_n < 1$ and $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. Define $\delta := 1 - \sum_{n=1}^{\infty} \gamma_n$ and next define $h_\lambda := (\gamma_1 + \frac{\lambda}{2}\delta) f_1 + (\gamma_2 + \frac{\lambda}{2}\delta) f_2 + (\gamma_3 + (1-\lambda)\delta) f_3 + \sum_{n=4}^{\infty} \gamma_n f_n$.

We want $h_\lambda$ to be in $E$, so we restrict values of $\lambda$ to be in $\left[-\frac{\gamma_1}{\delta},\ \frac{\gamma_3}{\delta}+1\right]$. Then

$$\begin{aligned}
|||h_\lambda - z||| &= \left|\left|\left|\left(\frac{\lambda}{2}b\delta, \frac{\lambda}{2}b\delta, (1-\lambda)\delta b, 0, 0, \cdots\right)\right|\right|\right| \\
&= \max\begin{cases} (\alpha+\beta)(1-\lambda)b\delta - \beta\lambda b\delta & \text{if } \lambda \in \left[-\frac{\gamma_1}{\delta},\ 0\right), \\ (\alpha+\beta)(1-\lambda)b\delta + \beta\lambda b\delta & \text{if } \lambda \in \left[0,\ \frac{2}{3}\right), \\ \beta b\delta + \frac{\alpha b\delta\lambda}{2} & \text{if } \lambda \in \left[\frac{2}{3},\ 1\right), \\ \frac{(\alpha+4\beta)b\delta\lambda}{2} - \beta b\delta & \text{if } \lambda \in \left[1,\ \frac{\gamma_3}{\delta}+1\right]. \end{cases}
\end{aligned}$$

Define $\Gamma := \min_{\lambda \in \left[-\frac{\gamma_1}{\delta},\ \frac{\gamma_2}{\delta}+1\right]} |||h_\lambda - z|||$. Therefore, $|||h_\lambda - z|||$ is minimized when $\lambda \in [0,1]$ with unique minimizer such that its minimum value would be $\Gamma = \frac{(\alpha+3\beta)b\delta}{3}$.

Now fix $y \in E$ of the form $\sum_{n=1}^{\infty} t_n f_n$ such that $\sum_{n=1}^{\infty} t_n = 1$ with $t_n \geq 0$, $\forall n \in \mathbb{N}$.

Then,

$$\begin{aligned}
|||y - z||| &= \left|\left|\left|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right|\right|\right| \\
&= |||(t_1-\gamma_1)be_1 + (t_2-\gamma_2)be_2 + (t_3-\gamma_3)be_3 + (t_4-\gamma_4)e_4 + \cdots||| \\
&= \max\left\{\begin{array}{l} (\alpha+\beta)b|t_1-\gamma_1| + \beta b|t_2-\gamma_2| + \beta b|t_3-\gamma_3| + \beta|t_4-\gamma_4| + \beta|t_5-\gamma_5| + \cdots, \\ (\alpha+\beta)b|t_2-\gamma_2| + \beta b|t_1-\gamma_1| + \beta b|t_3-\gamma_3| + \beta|t_4-\gamma_4| + \beta|t_5-\gamma_5| + \cdots, \\ (\alpha+\beta)b|t_3-\gamma_3| + \beta b|t_1-\gamma_1| + \beta b|t_2-\gamma_2| + \beta|t_4-\gamma_4| + \beta|t_5-\gamma_5| + \cdots, \\ (\alpha+\beta)|t_4-\gamma_4| + \beta b|t_1-\gamma_1| + \beta b|t_2-\gamma_2| + \beta b|t_3-\gamma_3| + \beta|t_5-\gamma_5| + \cdots, \\ (\alpha+\beta)|t_5-\gamma_5| + \beta b|t_1-\gamma_1| + \beta b|t_2-\gamma_2| + \beta b|t_3-\gamma_3| + \beta|t_4-\gamma_4| \\ +\beta|t_6-\gamma_6| + \cdots, \\ \cdots\cdots\cdots\cdots \end{array}\right\}.
\end{aligned}$$

*Subcase 2.1:* $|t_1-\gamma_1| \geq |t_2-\gamma_2|$, $|t_1-\gamma_1| \geq |t_3-\gamma_3|$ and $b|t_1-\gamma_1| \geq |t_k-\gamma_k|$, $\forall k \geq 4$.
Then,

$$\begin{aligned}
|||y - z||| &\geq \beta b\delta + \alpha b|t_1-\gamma_1| + \beta(1-b)\sum_{k=4}^{\infty}|t_k-\gamma_k| \\
&\geq \beta b\delta + \alpha b|t_1-\gamma_1| + \beta(1-b)|\delta - (t_1-\gamma_1) - (t_2-\gamma_2)| \\
&\geq \beta\delta + [(2b-2)\beta + b\alpha]|t_1-\gamma_1|.
\end{aligned}$$

*Subcase 2.1.1:* Assume $\frac{\delta}{3} \geq |t_1-\gamma_1|$.

Then clearly the last inequality from above says that $|||y-z||| \geq \frac{(\alpha+3\beta)b\delta}{3}$.

*Subcase 2.1.2:* Assume $\frac{\delta}{3} < |t_1-\gamma_1|$.

Then $|||y-z||| \geq \beta b\delta + \alpha b|t_1-\gamma_1| + \beta(1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \geq \frac{(\alpha+3\beta)b\delta}{3}$.

*Subcase 2.2:* $|t_2-\gamma_2| \geq |t_1-\gamma_1|$, $|t_2-\gamma_2| \geq |t_3-\gamma_3|$ and $b|t_2-\gamma_2| \geq |t_k-\gamma_k|$, $\forall k \geq 4$.
Then,

$$\begin{aligned}
|||y - z||| &\geq \beta b\delta + \alpha b|t_2-\gamma_2| + \beta(1-b)\sum_{k=4}^{\infty}|t_k-\gamma_k| \\
&\geq \beta b\delta + \alpha b|t_2-\gamma_2| + \beta(1-b)|\delta - (t_1-\gamma_1) - (t_2-\gamma_2)| \\
&\geq \beta\delta + [(2b-2)\beta + b\alpha]|t_2-\gamma_2|.
\end{aligned}$$

*Subcase 2.2.1:* Assume $\frac{\delta}{3} \geq |t_2-\gamma_2|$.

Then clearly the last inequality from above says that $|||y-z||| \geq \frac{(\alpha+3\beta)b\delta}{3}$.

*Subcase 2.2.2:* Assume $\frac{\delta}{3} < |t_2-\gamma_2|$.

Then $\|\|y-z\|\| \geq \beta b\delta + \alpha b|t_2-\gamma_2| + \beta(1-b)\sum\limits_{k=3}^{\infty}|t_k-\gamma_k| \geq \frac{(\alpha+3\beta)b\delta}{3}$.

*Subcase 2.3:* $|t_3-\gamma_3| \geq |t_1-\gamma_1|$, $|t_3-\gamma_3| \geq |t_2-\gamma_2|$ and $b|t_3-\gamma_3| \geq |t_k-\gamma_k|$, $\forall k \geq 4$.

Then,

$$\begin{aligned} \|\|y-z\|\| &\geq \beta b\delta + \alpha b|t_3-\gamma_3| + \beta(1-b)\sum_{k=4}^{\infty}|t_k-\gamma_k| \\ &\geq \beta b\delta + \alpha b|t_3-\gamma_3| + (1-b)|\delta-(t_1-\gamma_1)-(t_2-\gamma_2)-(t_3-\gamma_3)| \\ &\geq \beta\delta + [(3b-3)\beta + b\alpha]|t_3-\gamma_3|. \end{aligned}$$

*Subcase 2.3.1:* Assume $\frac{\delta}{3} \geq |t_3-\gamma_3|$.

Then clearly the last inequality from above says that $\|\|y-z\|\| \geq \frac{(\alpha+3\beta)b\delta}{3}$.

*Subcase 2.3.2:* Assume $\frac{\delta}{3} < |t_3-\gamma_3|$.

Then $\|\|y-z\|\| \geq \beta b\delta + \alpha b|t_3-\gamma_3| + \beta(1-b)\sum\limits_{k=4}^{\infty}|t_k-\gamma_k| \geq \frac{(\alpha+3\beta)b\delta}{3}$.

*Subcase 2.4:* $|t_4-\gamma_4| \geq b|t_1-\gamma_1|$, $|t_4-\gamma_4| \geq b|t_2-\gamma_2|$, $|t_4-\gamma_4| \geq b|t_3-\gamma_3|$ and $|t_4-\gamma_4| \geq |t_k-\gamma_k|$, $\forall k \geq 5$.

Then,

$$\begin{aligned} \|\|y-z\|\| &\geq \beta b\delta + \alpha|t_4-\gamma_4| + \beta(1-b)\sum_{k=4}^{\infty}|t_k-\gamma_k| \\ &\geq \beta b\delta + \alpha|t_4-\gamma_4| + \beta(1-b)|\delta-(t_1-\gamma_1)-(t_2-\gamma_2)-(t_3-\gamma_3)| \\ &\geq \beta b\delta + \alpha|t_4-\gamma_4| + \beta(1-b)\delta - \frac{3\beta(1-b)}{b}|t_4-\gamma_4| \\ &\geq \beta\delta + \frac{(3b-3)\beta + b\alpha}{b}|t_4-\gamma_4|. \end{aligned}$$

*Subcase 2.4.1:* Assume $\frac{b\delta}{3} \geq |t_3-\gamma_3|$.

Then clearly the last inequality from above says that $\|\|y-z\|\| \geq \frac{(\alpha+3\beta)b\delta}{3}$.

*Subcase 2.4.2:* Assume $\frac{b\delta}{3} < |t_4-\gamma_4|$.

Then $\|\|y-z\|\| \geq \beta b\delta + \alpha|t_4-\gamma_4| + \beta(1-b)\sum\limits_{k=4}^{\infty}|t_k-\gamma_k| \geq \frac{(\alpha+3\beta)b\delta}{3}$.

Thus, we continue in this way and see that $\|\|y-z\|\| \geq \frac{(\alpha+3\beta)b\delta}{3}$ from all cases.

Therefore, when $\lambda$ is choosen to be in $[0,1]$, for any $y \in E$ and for $z \in W \setminus E$, $\|\|y-z\|\| \geq \Gamma$. Then the rest follows as in the proof of Theorem 3.2.

□

Now we can give the generalized results with their proofs by just noting what the difference from those of previous theorems would be.

**Corollary 3.1.** *Fix an integer $N > 3$ and $b \in \left(0, \frac{2}{3}\right)$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in $c_0$ by setting $f_1 := b\,e_1$, $f_2 := b\,e_2$, $f_3 := b\,e_3, \cdots, f_N := b\,e_N$ and $f_n := e_n$, for all integers $n \geq N+1$. Next, define the closed, bounded, convex subset $E = E_b$ of $\ell^1$ by $E := \left\{\sum\limits_{n=1}^{\infty} t_n\, f_n :\ each\ t_n \geq 0\ and\ \sum\limits_{n=1}^{\infty} t_n = 1\right\}$. Then, $E$ has the fixed point property for affine $\|\|.\|\|$-nonexpansive mappings.*

*Proof.* We would use exactly similar strategy in the proof of the previous theorems and in Case 2 we would have the following changes followed by the necessary steps as in the proof of the previous theorems. Define $\delta := 1 - \sum_{n=1}^{\infty}\gamma_n$ and next define $h_\lambda :=$

$(\gamma_1+\frac{\lambda}{N-1}\delta)f_1+(\gamma_2+\frac{\lambda}{N-1}\delta)f_2+\cdots+(\gamma_{N-1}+\frac{\lambda}{N-1}\delta)f_{N-1}+(\gamma_N+(1-\lambda)\delta)f_N+\sum_{n=N+1}^{\infty}\gamma_n f_n$. Then, we would obtain that $|||h_\lambda - z|||$ is minimized when $\lambda \in [0,1]$ with unique minimizer such that its minimum value would be $\Gamma = \frac{(\alpha+N\beta)b\delta}{N}$. $\square$

## 4. Conclusions

In our presented investigation, we have systematically studied large classes of non-weak* compact, closed, bounded subsets in a generalized degenerate Lorentz-Marcinkiewicz . We have generalized the classes obtained in the studies [12, 14, 15] and showed that there exists large classes of non-weak* compact, closed, convex and bounded sets with the fixed point property for affine nonexpansive mappings in the generalized degenerate Lorentz-Marcinkiewicz space.

## References


[1] Astashkin, S.V., and Sukochev, F.A., (2007), Banach-Saks property in Marcinkiewicz spaces, J. Math. Anal. Appl., 336, pp. 1231-1258.

[2] Aasma, A, Dutta, H. and Natarajan, P.N., (2017), An Introductory Course in Summability Theory, 1st ed., John Wiley & Sons, Inc. Hoboken, USA

[3] Dutta, H. and Rhoades, B.E. (Eds.), (2016), Current Topics in Summability Theory and Applications, 1st ed., Springer, Singapore.

[4] Delibaş, M., (2019), Fixed point properties for nonexpansive mappings in degenerate Lorentz-Marcinkiewicz spaces, M.Sc. thesis, Kafkas University, Kars, Turkey.

[5] Everest, T., (2013), Fixed points of nonexpansive maps on closed, bounded, convex sets in $l^1$, Ph.D., University of Pittsburgh, Pittsburgh, PA, USA.

[6] Goebel, K. and Kuczumow, T., (1979), Irregular convex sets with fixed-point property for nonexpansive mappings. Colloq. Math., 40, pp. 259-264.

[7] Kaczor, W. and Prus, S., (2004), Fixed point properties of some sets in l1. Proceedings of the International Conference on Fixed Point Theory and Applications, 11p.

[8] Lin, P. K., (2008), There is an equivalent norm on $\ell_1$ that has the fixed point property, Nonlinear Anal., 68, pp. 2303-2308.

[9] Lindenstrauss, J. and Tzafriri, L., (1977), Classical Banach spaces I: sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, 92, Springer-Verlag.

[10] Lorentz, G.G., (1950), Some new functional spaces. Ann. Math., pp. 37-55.

[11] Nezir, V., (2012), Fixed point properties for $c_0$-like spaces, Ph.D. thesis, University of Pittsburgh, Pittsburgh, PA, USA.

[12] Nezir, V., (2019), Fixed point properties for a degenerate Lorentz-Marcinkiewicz space, Turk. J. Math., 43, pp. 1919-1939.

[13] Nezir, V., (2019), Fixed point properties for Lorentz sequence spaces $\ell^0_{\rho,\infty}$ and $\ell_{\rho,1}$, J. Fixed Point Theory Appl., to appear.

[14] Nezir, V. and Mustafa, N., (2018), On the fixed point property for a degenerate Lorentz-Marcinkiewicz Space, Proceedings of the 5th international conference on recent advances in pure and applied mathematics (Icrapam 2018), Karadeniz Technical University, Trabzon, Turkey, pp. 45-51.

[15] Nezir, V., Mustafa, N. and Delibaş, M., (2018), Infinite dimensional subspaces of a degenerate Lorentz-Marcinkiewicz space with the fixed point property, Proceedings of the 5th international conference on recent advances in pure and applied mathematics (Icrapam 2018), Karadeniz Technical University, Trabzon, Turkey, pp. 45-51.

[16] Rakov, S.A., (1979), Banach-Saks property of a Banach space, Mathematical notes of the Academy of Sciences of the USSR, 26, pp. 909-916.

[17] Rakov, S.A., (1982), Banach-Saks exponent of certain Banach spaces of sequences, Mathematical Notes of the Academy of Sciences of the USSR, 32, pp. 791-797.

[18] Schauder, J., (1930), Der Fixpunktsatz in Funktionalraümen, Studia Math., 2, pp. 171-180 (in German).

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**Hemen Dutta** for the photo and short autnobiography, see TWMS J. of Appl. and Engin. Math., current issue.

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