Available online: September 22, 2017

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 2, Pages 99–115 (2018)
DOI: 10.1501/Commua1_0000000865
ISSN 1303–5991

http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS SERIES A1

# THE RECURRENCE SEQUENCES VIA POLYHEDRAL GROUPS

ÖMÜR DEVECI, YEŞIM AKÜZÜM, AND COLIN M. CAMPBELL

Abstract. In this paper, we define recurrence sequences by using the relation matrices of the finite polyhedral groups and then, we obtain some of their properties. Also, we obtain the cyclic groups and the semigroups which are produced by the generating matrices when read modulo $\alpha$ and we study the sequences defined modulo $\alpha$. Then we derive the relationships between the orders of the cyclic groups obtained and the periods of the sequences defined working modulo $\alpha$. Furthermore, we extend these sequences to groups and obtain the periods of the sequences extended in the finite polyhedral groups case.

## 1. Introduction

The polyhedral group $(p, q, r)$ for $p, q, r > 1$, is defined by the presentation

$$\langle x, y, z \mid x^p = y^q = z^r = xyz = e\rangle$$

or

$$\langle x, y \mid x^p = y^q = (xy)^r = e\rangle.$$

The polyhedral group $(p, q, r)$ is finite if and only if the number

$$k = pqr\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1\right) = qr + rp + pq - pqr$$

is positive, i.e., in the case $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. Its order is $2pqr/k$. Using Tietze transformations we may show that $(p, q, r) \approxeq (q, r, p) \approxeq (r, p, q)$.

For more information on these groups, see [4].

Let $G$ be a finite $j$-generator group and let

$$X = \left\{(x_1, x_2, \ldots, x_j) \in \underbrace{G \times G \times \cdots \times G}_{j} \mid \langle x_1, x_2, \ldots, x_j\rangle = G\right\}.$$

Received by the editors: August 24, 2016; Accepted: June 12, 2017.

2010 *Mathematics Subject Classification.* 11B50; 20F05; 11C20; 20D60.

*Key words and phrases.* Matrix, polyhedral sequences, polyhedral groups, period.

We call $(x_1, x_2, \ldots, x_j)$ a generating $j$-tuple for $G$.

Let $G$ be the group defined by the finite presentation

$$G = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle.$$

The relation matrix of $G$ is an $m \times n$ matrix where the $(i,j)^{\text{th}}$ entry of the matrix is the sum of the exponents of the generator $x_j$ in the relator $r_i$.

For detailed information about the relation matrix, see [12].

**Example 1.1.** *The relation matrix of the group defined by the presentation* $\langle x, y, z \mid x^m = y^2 = z^2 = xyz = e \rangle$ *is*

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding $k$ terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [13], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix $A$ be defined by

$$A = [a_{ij}]_{k \times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied by many authors [2, 5, 6, 10, 11, 13, 15, 19, 20, 21, 22]. In Section 2, we develop properties of the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind which are obtained from the matices defined by the aid of the relation matrices of the polyhedral groups $(m,2,2)$, $(2,m,2)$, $(2,2,m)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$.

In [5, 6, 7, 17], the authors have produced the cyclic groups and the semigroups via some special matrices and then, they have studied the orders of these algebraic structures. In Section 3, we obtain the cyclic groups and the semigroups by using the generating matrices of the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind when read modulo $\alpha$ and then, we give their miscellaneous properties.

The study of recurrence sequences in groups began with the earlier work of Wall [23] where the ordinary Fibonacci sequences in cyclic groups has been investigated. In the mid eighties, Wilcox extended the problem to abelian groups [24]. Further, the theory has been expanded to some special linear recurrence sequences by several authors; see, for example, [1, 3, 5, 6, 8, 9, 14, 16]. In Section 3, we study the defined sequences modulo $\alpha$ and then, we derive the relationships among the orders of the cyclic groups obtained and the periods of these sequences. Also, in this section, we redefine the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind by means of the elements of the groups which have two or three generators and then, we examine these sequences in the finite groups. Finally, we obtain the lengths of the periods of the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind in the polyhedral groups $(m, 2, 2)$, $(2, m, 2)$, $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ by using the periods of these sequences with respect to a modulus $\alpha$, where we consider each one of the sequences in one group such that the sequence is produced by the aid of the presentation of this group.

## 2. Polyhedral Sequences

We next define the matrices $M_1$, $M_2$, $M_3$, $M_4$, $M_5$ and $M_6$ by using the presentations of the polyhedral groups $(m, 2, 2)$, $(2, m, 2)$, $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ in the two generator cases, that is for generating pair $(x, y)$, as follows, respectively:

$$M_u = \begin{bmatrix} \alpha_1 & 0 & 1 \\ 0 & \alpha_2 & 1 \\ \alpha_3 & \alpha_3 & 1 \end{bmatrix}, \ (u = 1, 2, 3,\ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u)$$

and

$$M_v = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ v-1 & v-1 & 1 \end{bmatrix}, \ (v = 4, 5, 6)\,.$$

Similarly, we define the matrices $M_1^*$, $M_2^*$, $M_3^*$, $M_4^*$, $M_5^*$ and $M_6^*$ by the aid of the presentations of these groups in the three generator cases, that is for generating triple $(x, y, z)$, as follows, respectively:

$$M_u^* = \begin{bmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_2 & 0 & 1 \\ 0 & 0 & \alpha_3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \ (u = 1, 2, 3, \ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u).$$

and

$$M_v^* = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & v-1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \ (v = 4, 5, 6).$$

Note that $\det M_1 = \det M_2 = -4$, $\det M_3 = 4-4m$, $\det M_4 = -9$, $\det M_5 = -14$, $\det M_6 = -19$, $\det M_1^* = \det M_2^* = \det M_3^* = -4$, $\det M_4^* = -3$, $\det M_5^* = -2$ and $\det M_6^* = -1$.

We now define new sequences from the matrices $M_k$ and $M_k^*$, $(k = 1, \ldots, 6)$ as shown, respectively:

$$a_n^u = \begin{cases} a_{n-1}^u + \alpha_1 a_{n-3}^u & n \equiv 1 \,(\mathrm{mod}\, 3), \\ a_{n-2}^u + \alpha_2 a_{n-3}^u & n \equiv 2 \,(\mathrm{mod}\, 3), \\ a_{n-3}^u + \alpha_3 a_{n-4}^u + \alpha_3 a_{n-5}^u & n \equiv 0 \,(\mathrm{mod}\, 3), \end{cases}$$
$$(u = 1, 2, 3, \ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u),$$

$$a_n^v = \begin{cases} a_{n-1}^v + 2a_{n-3}^v & n \equiv 1 \,(\mathrm{mod}\, 3), \\ a_{n-2}^v + 3a_{n-3}^v & n \equiv 2 \,(\mathrm{mod}\, 3), \\ a_{n-3}^v + (v-1)\, a_{n-4}^v + (v-1)\, a_{n-5}^v & n \equiv 0 \,(\mathrm{mod}\, 3), \end{cases}$$
$$(v = 4, 5, 6)$$

for $n \geq 4$, where $a_1^k = 0$, $a_2^k = 0$, $a_3^k = 1$ and

$$b_n^u = \begin{cases} b_{n-1}^u + \alpha_1 b_{n-4}^u & n \equiv 1 \,(\mathrm{mod}\, 4), \\ b_{n-2}^u + \alpha_2 b_{n-4}^u & n \equiv 2 \,(\mathrm{mod}\, 4), \\ b_{n-3}^u + \alpha_3 b_{n-4}^u & n \equiv 3 \,(\mathrm{mod}\, 4), \\ b_{n-4}^u + b_{n-5}^u + b_{n-6}^u + b_{n-7}^u & n \equiv 0 \,(\mathrm{mod}\, 4), \end{cases}$$
$$(u = 1, 2, 3, \ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u),$$

$$b_n^v = \begin{cases} b_{n-1}^v + 2b_{n-4}^v & n \equiv 1 \,(\mathrm{mod}\, 4), \\ b_{n-2}^v + 3b_{n-4}^v & n \equiv 2 \,(\mathrm{mod}\, 4), \\ b_{n-3}^v + (v-1)\, b_{n-4}^v & n \equiv 3 \,(\mathrm{mod}\, 4), \\ b_{n-4}^v + b_{n-5}^v + b_{n-6}^v + b_{n-7}^v & n \equiv 0 \,(\mathrm{mod}\, 4), \end{cases} \quad (v = 4, 5, 6)$$

for $n \geq 5$, where $b_1^k = 0$, $b_2^k = 0$, $b_3^k = 0$, $b_4^k = 1$.

The sequences $\{a_n^k\}$ and $\{b_n^k\}$ for $k = 1, \ldots, 6$ are called the 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind, respectively.

By an inductive argument for $n \geq 3$ and $k = 1, \ldots, 6$, we may write

$$(M_k)^n = \begin{bmatrix} m_1^k & m_2^k & a_{3n+1}^k \\ m_3^k & m_4^k & a_{3n+2}^k \\ \lambda_k a_{3n+1}^k & \lambda_k a_{3n+2}^k & a_{3n+3}^k \end{bmatrix},$$
$$(\lambda_1 = \lambda_2 = 2,\ \lambda_3 = m,\ \lambda_4 = 3,\ \lambda_5 = 4,\ \lambda_6 = 5)$$

where

$$\begin{aligned}
m_1^1 &= 2a_{3n-2}^1 + ma_{3n+1}^1 + m^{n-1} + \sum_{i=0}^{n-3} m^{n-2-i} a_{8+3i}^1, \\
m_1^2 &= a_{3n+1}^2 + a_{3n+3}^2 - 2m^{n-2} - 2\sum_{i=0}^{n-3} m^{n-3-i} a_{7+3i}^2, \\
m_1^3 &= \frac{a_{3n+2}^3 + a_{3n+3}^3 + 2^n}{2}, \\
m_1^4 &= 7.2^{n-2} + 3\sum_{i=0}^{n-3} 2^{n-3-i} a_{7+3i}^4, \\
m_1^5 &= 2^{n+1} + \sum_{i=0}^{n-3} 2^{n-1-i} a_{7+3i}^5, \\
m_1^6 &= 9.2^{n-2} + 5\sum_{i=0}^{n-3} 2^{n-3-i} a_{7+3i}^6,
\end{aligned}$$

$$\begin{aligned}
m_2^1 &= 2m^{n-2} + 2\sum_{i=0}^{n-3} m^{n-3-i} a_{8+3i}^1, \\
m_2^2 &= 2m^{n-2} + 2\sum_{i=0}^{n-3} m^{n-3-i} a_{7+3i}^2, \\
m_2^3 &= \frac{a_{3n+2}^3 + a_{3n+3}^3 - 2^n}{2}, \\
m_2^4 &= 3^{n-1} + \sum_{i=0}^{n-3} 3^{n-2-i} a_{7+3i}^4, \\
m_2^5 &= 2^n + \sum_{i=0}^{n-3} 2^{n-1-i} a_{7+3i}^5, \\
m_2^6 &= 5 \cdot 3^{n-2} + 5\sum_{i=0}^{n-3} 3^{n-3-i} a_{7+3i}^6,
\end{aligned}$$

$m_3^1 = m_2^1$, $m_3^2 = m_2^2$, $m_3^3 = m_2^3$, $m_3^4 = m_2^4$, $m_3^5 = m_2^5$, $m_3^6 = m_2^6$

and

$$\begin{aligned}
m_4^1 &= a_{3n+2}^1 + a_{3n+3}^1 - 2m^{n-2} - 2\sum_{i=0}^{n-3} m^{n-3-i} a_{8+3i}^1,\\
m_4^2 &= 2a_{3n-1}^2 + ma_{3n+2}^2 - m^{n-1} - \sum_{i=0}^{n-3} m^{n-2-i} a_{7+3i}^2,\\
m_4^3 &= m_1^3,\\
m_4^4 &= 3^n + \sum_{i=0}^{n-2} 3^{n-1-i} a_{5+3i}^4,\\
m_4^5 &= 3^n + 4\sum_{i=0}^{n-2} 3^{n-2-i} a_{5+3i}^5,\\
m_4^6 &= 3^n + 5\sum_{i=0}^{n-2} 3^{n-2-i} a_{5+3i}^6.
\end{aligned}$$

Similarly, we obtain the matrices $(M_k^*)^n$ for $n \geq 3$ and $k = 1, \dots, 6$ by using mathematical induction as shown:

For $k = 1, 2, 3$,

$$(M_k^*)^n = \begin{bmatrix} m_1^{*k} & m_2^{*k} & m_3^{*k} & a_{4n+1}^{*k} \\ m_4^{*k} & m_5^{*k} & m_6^{*k} & a_{4n+2}^{*k} \\ m_7^{*k} & m_8^{*k} & m_9^{*k} & a_{4n+3}^{*k} \\ a_{4n+1}^{*k} & a_{4n+2}^{*k} & a_{4n+3}^{*k} & a_{4n+4}^{*k} \end{bmatrix},$$

where

$$\begin{aligned}
m_1^{*1} &= a_{4n-3}^{*1} + ma_{4n+1}^{*1} - m^{n-1} - \sum_{i=0}^{n-3} m^{n-2-i} a_{10+4i}^{*1},\ m_1^{*2} = a_{4n-3}^{*2} + 2^n\\
&\quad + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*2},\\
m_1^{*3} &= a_{4n-3}^{*3} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*3},
\end{aligned}$$

$$\begin{aligned}
m_2^{*1} &= m^{n-2} + \sum_{i=0}^{n-3} m^{n-3-i} a_{10+4i}^{*1},\ m_2^{*2} = m^{n-2} + \sum_{i=0}^{n-3} m^{n-3-i} a_{9+4i}^{*2},\ m_2^{*3} = a_{4n-3}^{*3}\\
&\quad + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*3},
\end{aligned}$$

$$m_3^{*1} = m_2^{*1},\, m_3^{*2} = a_{4n-3}^{*2} + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*2},\, m_3^{*3} = m^{n-2} + \sum_{i=0}^{n-3} m^{n-3-i} a_{9+4i}^{*3},$$

$$m_4^{*1} = m_2^{*1},\, m_4^{*2} = m_2^{*2},\, m_4^{*3} = m_2^{*3},$$

$$\begin{aligned} m_5^{*1} &= a_{4n-2}^{*1} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{6+4i}^{*1},\, m_5^{*2} \\ &= a_{4n-2}^{*2} + m a_{4n+2}^{*2} - m^{n-1} + \sum_{i=0}^{n-3} m^{n-2-i} a_{9+4i}^{*2},\, m_5^{*3} = m_1^{*3}, \end{aligned}$$

$$m_6^{*1} = a_{4n-2}^{*1} + \sum_{i=0}^{n-3} 2^{n-2-i} a_{6+4i}^{*1},\, m_6^{*2} = m_2^{*2},\, m_6^{*3} = m_3^{*3},$$

$$m_7^{*1} = m_2^{*1},\, m_7^{*2} = m_3^{*2},\, m_7^{*3} = m_3^{*3},$$

$$m_8^{*1} = m_6^{*1},\, m_8^{*2} = m_2^{*2},\, m_8^{*3} = m_3^{*3}$$

and

$$m_9^{*1} = m_5^{*1},\, m_9^{*2} = m_1^{*2},\, m_9^{*3} = a_{4n-1}^{*3} + m a_{4n+3}^{*3} - m^{n-1} + \sum_{i=0}^{n-3} m^{n-2-i} a_{9+4i}^{*3}.$$

For $k = 4, 5, 6$,

$$(M_4^*)^n = \begin{bmatrix} a_{4n-3}^{*4} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*4} & a_{4n+2}^{*4} - a_{4n+1}^{*4} & a_{4n+2}^{*4} - a_{4n+1}^{*4} & a_{4n+1}^{*4} \\ a_{4n+2}^{*4} - a_{4n+1}^{*4} & a_{4n-2}^{*4} + 3^n + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*4} & a_{4n-2}^{*4} + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*4} & a_{4n+2}^{*4} \\ a_{4n+2}^{*4} - a_{4n+1}^{*4} & a_{4n-2}^{*4} + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*4} & a_{4n-2}^{*4} + 3^n + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*4} & a_{4n+3}^{*4} \\ a_{4n+1}^{*4} & a_{4n+2}^{*4} & a_{4n+3}^{*4} & a_{4n+4}^{*4} \end{bmatrix},$$

$$(M_5^*)^n = \begin{bmatrix} a_{4n-3}^{*5} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*5} & a_{4n+2}^{*5} - a_{4n+1}^{*5} & a_{4n-3}^{*5} + \sum_{i=0}^{n-3} 4^{n-2-i} a_{5+4i}^{*5} & a_{4n+1}^{*5} \\ a_{4n+2}^{*5} - a_{4n+1}^{*5} & a_{4n-2}^{*5} + 3^n + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*5} & a_{4n+3}^{*5} - a_{4n+2}^{*5} & a_{4n+2}^{*5} \\ a_{4n-3}^{*5} + \sum_{i=0}^{n-3} 4^{n-2-i} a_{5+4i}^{*5} & a_{4n+3}^{*5} - a_{4n+2}^{*5} & a_{4n-1}^{*5} + 4^n + \sum_{i=0}^{n-3} 4^{n-2-i} a_{7+4i}^{*5} & a_{4n+3}^{*5} \\ a_{4n+1}^{*5} & a_{4n+2}^{*5} & a_{4n+3}^{*5} & a_{4n+4}^{*5} \end{bmatrix}$$

and

$$(M_6^*)^n = \begin{bmatrix} a_{4n-3}^{*6} + 2^n + \sum_{i=0}^{n-3} 2^{n-2-i} a_{5+4i}^{*6} & a_{4n+2}^{*6} - a_{4n+1}^{*6} & a_{4n-1}^{*6} + \sum_{i=0}^{n-3} 2^{n-2-i} a_{7+4i}^{*6} & a_{4n+1}^{*6} \\ a_{4n+2}^{*6} - a_{4n+1}^{*6} & a_{4n-2}^{*6} + 3^n + \sum_{i=0}^{n-3} 3^{n-2-i} a_{6+4i}^{*6} & a_{4n-2}^{*6} + \sum_{i=0}^{n-3} 5^{n-2-i} a_{6+4i}^{*6} & a_{4n+2}^{*6} \\ a_{4n-1}^{*6} + \sum_{i=0}^{n-3} 2^{n-2-i} a_{7+4i}^{*6} & a_{4n-2}^{*6} + \sum_{i=0}^{n-3} 5^{n-2-i} a_{6+4i}^{*6} & a_{4n-1}^{*6} + 5^n + \sum_{i=0}^{n-3} 5^{n-2-i} a_{7+4i}^{*6} & a_{4n+3}^{*6} \\ a_{4n+1}^{*6} & a_{4n+2}^{*6} & a_{4n+3}^{*6} & a_{4n+4}^{*6} \end{bmatrix}.$$

It is well-known that the Simpson formula for a recurrence sequence can be obtained from the determinant of its generating matrix. For example, the Simpson formula for the sequence $\{a_n^3\}$ is

$$\begin{aligned}(4-4m)^n &= \left(a_{3n+2}^3+a_{3n+3}^3+2^n\right)\left(-\frac{m}{2}\left(a_{3n+1}^3\right)^2-\frac{m}{2}\left(a_{3n+2}^3\right)^2\right)+\\ &\quad m a_{3n+1}^3\left(\left(a_{3n+2}^3\right)^2+a_{3n+2}^3a_{3n+3}^3-2^n a_{3n+2}^3\right)+2^n a_{3n+3}^3\left(a_{3n+3}^3+a_{3n+2}^3\right).\end{aligned}$$

It is easy to see that the characteristic equations of the sequences $\{a_n^k\}$ and $\{b_n^k\}$, $(k=1,\ldots,6)$ do not have multiple roots; that is, each of the eigenvalues of the matrices $M_k$ and $M_k^*$ is distinct.

Let $\{x_1^k,x_2^k,x_3^k\}$ and $\{x_1^k,x_2^k,x_3^k,x_4^k\}$ be the sets of the eigenvalues of the matrices $M_k$ and $M_k^*$ for $k=1,\ldots,6$, respectively and let $V_k^{(u)}$ be a $(u+2)\times(u+2)$ Vandermonde matrix as follows:

$$V_k^{(u)}=\begin{bmatrix}\left(x_1^k\right)^{u+1} & \left(x_2^k\right)^{u+1} & \cdots & \left(x_{u+2}^k\right)^{u+1}\\ \left(x_1^k\right)^{u} & \left(x_2^k\right)^{u} & \cdots & \left(x_{u+2}^k\right)^{u}\\ \vdots & \vdots & & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix}$$

where $u=1,2$. Suppose now that

$$W_k^i=\begin{bmatrix}\left(x_1^k\right)^{n+u+2-i}\\ \left(x_2^k\right)^{n+u+2-i}\\ \vdots\\ \left(x_{u+2}^k\right)^{n+u+2-i}\end{bmatrix}$$

and $V_{k,j}^{(u,i)}$ İS a $(u+2)\times(u+2)$ matrix obtained from $V_k^{(u)}$ by replacing the $j$th column of $V_k^{(u)}$ by $W_k^i$. This yields the Binet-type formulas for the sequences $\{a_n^k\}$ and $\{b_n^k\}$, namely.

**Theorem 2.1.** *For $k=1,\ldots,6$,*

$$m_{ij}^{(k,n)}=\frac{\det V_{k,j}^{(1,i)}}{\det V_k^{(1)}} \text{ and } m_{ij}^{*(k,n)}=\frac{\det V_{k,j}^{(2,i)}}{\det V_k^{(2)}},$$

*where $(M_k)^n=m_{ij}^{(k,n)}$ and $(M_k^*)^n=m_{ij}^{*(k,n)}$.*

*Proof.* Since the eigenvalues of the matrices $M_k$ and $M_k^*$ are are distinct, these matrices are diagonalizable. Let

$$D^{(1,k)}=\operatorname{diag}\left(x_1^k,x_2^k,x_3^k\right) \text{ and } D^{(2,k)}=\operatorname{diag}\left(x_1^k,x_2^k,x_3^k,x_4^k\right),$$

then it is easy to see that $M_kV_k^{(1)}=V_k^{(1)}D^{(1,k)}$ and $M_k^*V_k^{(2)}=V_k^{(2)}D^{(2,k)}$. Since the matrices $V_k^{(1)}$ and $V_k^{(2)}$ are invertible, $\left(V_k^{(1)}\right)^{-1}M_kV_k^{(1)}=D^{(1,k)}$ and $\left(V_k^{(2)}\right)^{-1}M_k^*V_k^{(2)}=$

$D^{(2,k)}$. Thus, the matrices $M_k$ and $M_k^*$ are similar to $D^{(1,k)}$ and $D^{(2,k)}$, respectively. So, we get $(M_k)^n V_k^{(1)} = V_k^{(1)} \left(D^{(1,k)}\right)^n$ and $(M_k^*)^n V_k^{(2)} = V_k^{(2)} \left(D^{(2,k)}\right)^n$ for $n \geq 1$.

Then we can write the following linear system of equations:

$$\begin{cases} m_{i1}^{(k,n)} \left(x_1^k\right)^2 + m_{i2}^{(k,n)} \left(x_1^k\right) + m_{i3}^{(k,n)} = \left(x_1^k\right)^{n+3-i} \\ m_{i1}^{(k,n)} \left(x_2^k\right)^2 + m_{i2}^{(k,n)} \left(x_2^k\right) + m_{i3}^{(k,n)} = \left(x_2^k\right)^{n+3-i} \\ m_{i1}^{(k,n)} \left(x_3^k\right)^2 + m_{i2}^{(k,n)} \left(x_3^k\right) + m_{i3}^{(k,n)} = \left(x_3^k\right)^{n+3-i} \end{cases}, \ (1 \leq i, j \leq 3)$$

and

$$\begin{cases} m_{i1}^{*(k,n)} \left(x_1^k\right)^3 + m_{i2}^{*(k,n)} \left(x_1^k\right)^2 + m_{i3}^{*(k,n)} \left(x_1^k\right) + m_{i4}^{*(k,n)} = \left(x_1^k\right)^{n+4-i} \\ m_{i1}^{*(k,n)} \left(x_2^k\right)^3 + m_{i2}^{*(k,n)} \left(x_2^k\right)^2 + m_{i3}^{*(k,n)} \left(x_2^k\right) + m_{i4}^{*(k,n)} = \left(x_2^k\right)^{n+4-i} \\ m_{i1}^{*(k,n)} \left(x_3^k\right)^3 + m_{i2}^{*(k,n)} \left(x_3^k\right)^2 + m_{i3}^{*(k,n)} \left(x_3^k\right) + m_{i4}^{*(k,n)} = \left(x_3^k\right)^{n+4-i} \\ m_{i1}^{*(k,n)} \left(x_4^k\right)^3 + m_{i2}^{*(k,n)} \left(x_4^k\right)^2 + m_{i3}^{*(k,n)} \left(x_4^k\right) + m_{i4}^{*(k,n)} = \left(x_4^k\right)^{n+4-i} \end{cases}, \ (1 \leq i, j \leq 4).$$

Therefore, we obtain

$$m_{ij}^{(k,n)} = \frac{\det V_{k,j}^{(1,i)}}{\det V_k^{(1)}} \text{ and } m_{ij}^{*(k,n)} = \frac{\det V_{k,j}^{(2,i)}}{\det V_k^{(2)}}$$

for $k = 1, \ldots, 6$. □

## 3. The Cyclic Groups and The Semigroups via The Matrices $M_k$ and $M_k^*$

Given an integer matrix $A = [a_{ij}]$, $A \,(\mathrm{mod}\, \alpha)$ means that all entries of $A$ are modulo $\alpha$, that is, $A \,(\mathrm{mod}\, \alpha) = (a_{ij} \,(\mathrm{mod}\, \alpha))$. Let us consider the set $\langle A \rangle_\alpha = \left\{A^i \,(\mathrm{mod}\, \alpha) \mid i \geq 0\right\}$. If $\gcd(\alpha, \det A) = 1$, then $\langle A \rangle_\alpha$ is a cyclic group; if $\gcd(\alpha, \det A) \neq 1$, then $\langle A \rangle_\alpha$ is a semigroup. Let the notation $|\langle A \rangle_\alpha|$ denote the order of the set $\langle A \rangle_\alpha$.

We next consider the orders of the cyclic groups and the semigroups generated by the matrices $M_k$ and $M_k^*$ for $k = 1, \ldots, 6$.

**Theorem 3.1.** *Let $p$ be a prime and let $\langle G \rangle_{p^n}$ be any of the cyclic groups of $\langle M_k \rangle_{p^n}$ and $\langle M_k^* \rangle_{p^n}$ for $k = 1, \ldots, 6$ and $n \in N$. If $i$ is the largest positive integer such that $\left|\langle G \rangle_{p^i}\right| = \left|\langle G \rangle_p\right|$, then $\left|\langle G \rangle_{p^j}\right| = p^{j-i} \left|\langle G \rangle_p\right|$. In particular, if $\left|\langle G \rangle_{p^2}\right| \neq \left|\langle G \rangle_p\right|$, then $\left|\langle G \rangle_{p^j}\right| = p^{j-1} \left|\langle G \rangle_p\right|$.*

*Proof.* Let us consider the cyclic group $\langle M_1 \rangle_{p^n}$. Then $\gcd(p, 4) = 1$ that is, $p$ is an odd prime. Suppose that $u$ is positive integer and $\left|\langle M_1 \rangle_{p^n}\right|$ is denoted by $\circ(p^n)$. Since $(M_1)^{\circ\left(p^{u+1}\right)} \equiv I \left(\mathrm{mod}\, p^{u+1}\right)$, $(M_1)^{\circ\left(p^{u+1}\right)} \equiv I \,(\mathrm{mod}\, p^u)$ where $I$ is a $3 \times 3$

identity matrix. Thus, we show that $\circ(p^u)$ divides $\circ\left(p^{u+1}\right)$. Furthermore, if we denote

$$(M_1)^{\circ(p^u)} = I + \left(m_{ij}^{(u)} \cdot p^u\right),$$

then by the binomial expansion, we have

$$(M_1)^{\circ(p^u)\cdot p} = \left(I + \left(m_{ij}^{(u)} \cdot p^u\right)\right)^p = \sum_{r=0}^{p} \binom{p}{r} \left(m_{ij}^{(u)} \cdot p^u\right)^r \equiv I \left(\operatorname{mod} p^u\right).$$

So we get that $\circ\left(p^{u+1}\right)$ is divisible by $\circ\left(p^{u+1}\right) \cdot p$. Then, $\circ\left(p^{u+1}\right) = \circ(p^u)$ or $\circ\left(p^{u+1}\right) = \circ\left(p^{u+1}\right) \cdot p$. It is clear that the latter holds if and only if there exists an integer $m_{ij}^{(u)}$ which is not divisible by $p$. Since $i$ is the largest positive integer such that $\circ\left(p^i\right) = \circ(p)$ we have $\circ\left(p^{i+1}\right) \neq \circ\left(p^i\right)$, which yields that there exists an integer $m_{ij}^{(u)}$ such that $p \nmid m_{ij}^{(u)}$. So we find that $\circ\left(p^{i+2}\right) \neq \circ\left(p^{i+1}\right)$. To complete the proof we use an inductive method on $i$.

There are similar proofs for the other cyclic groups which are obtained as the above. $\square$

**Theorem 3.2.** *Let $\alpha$ be an positive integer and let $\langle G\rangle_\alpha$ be any of the cyclic groups of $\langle M_k\rangle_\alpha$ and $\langle M_k^*\rangle_\alpha$ for $k = 1, \ldots, 6$. If $\alpha$ has the prime factorization $\alpha = \prod_{j=1}^{t} p_j^{e_j}$, $(t \geq 1)$, then*

$$\left|\langle G\rangle_\alpha\right| = lcm\left[\left|\langle G\rangle_{p_1^{e_1}}\right|, \left|\langle G\rangle_{p_2^{e_2}}\right|, \ldots, \left|\langle G\rangle_{p_t^{e_t}}\right|\right].$$

*Proof.* Let us consider the cyclic group $\langle M_4^*\rangle_\alpha$, then $\gcd(\alpha, 3) = 1$. Suppose that $\left|\langle M_4^*\rangle_{p_j^{e_j}}\right| = v_j$ for $j = 1, \ldots, t$ and $\left|\langle M_4^*\rangle_\alpha\right| = v$. Then by $(M_4^*)^n$, we can write

$$\begin{aligned}
a_{4v_j-3}^{*4} + 2^{v_j} + \sum_{i=0}^{v_j-3} 2^{v_j-2-i} a_{5+4i}^{*4} &\equiv a_{4v_j-2}^{*4} + 3^{v_j} + \sum_{i=0}^{v_j-3} 3^{v_j-2-i} a_{6+4i}^{*4} \equiv a_{4v_j+4}^{*4} \\
&\equiv 1 \left(\operatorname{mod} p_j^{ej}\right), \\
a_{4v_j-2}^{*4} + \sum_{i=0}^{v_j-3} 3^{v_j-2-i} a_{6+4i}^{*4} &\equiv a_{4v_j+1}^{*4} \equiv a_{4v_j+2}^{*4} \equiv a_{4v_j+3}^{*4} \equiv 0 \left(\operatorname{mod} p_j^{e_j}\right)
\end{aligned}$$

and

$$\begin{aligned}
a_{4v-3}^{*4} + 2^{v} + \sum_{i=0}^{v-3} 2^{v-2-i} a_{5+4i}^{*4} &\equiv a_{4v-2}^{*4} + 3^{v} + \sum_{i=0}^{v-3} 3^{v-2-i} a_{6+4i}^{*4} \equiv a_{4v+4}^{*4} \equiv 1 \left(\operatorname{mod} \alpha\right), \\
a_{4v-2}^{*4} + \sum_{i=0}^{v-3} 3^{v-2-i} a_{6+4i}^{*4} &\equiv a_{4v+1}^{*4} \equiv a_{4v+2}^{*4} \equiv a_{4v+3}^{*4} \equiv 0 \left(\operatorname{mod} \alpha\right).
\end{aligned}$$

This implies that $(M_4^*)^v$ is of the form $\lambda \cdot (M_4^*)^{v_j}$, $(\lambda \in N)$ for all values of $j$. Thus it is verified that $v = \mathrm{lcm}[v_1, v_2, \ldots, v_t]$.

There are similar proofs for the other cyclic groups which are obtained as the above. □

We have the following useful results for the orders of the semigroups generated by the matrices $M_k$ and $M_k^*$ from $(M_k)^n$ and $(M_k^*)^n$ .

**Corollary 3.3.** *Let $\alpha = 2^\eta$ and $m = 2^\mu$ such that $\eta, \mu \in N$ and $1 \leq \mu \leq \eta$. Then the orders of the semigroups $\langle M_k \rangle_\alpha$ for $k = 1, 2, 3$ are as follows:*

*(i). If $\eta = \mu = 1$, then $|\langle M_k \rangle_\alpha| = 1$.*

*(ii). If $\eta \geq 2$ and $\mu = \eta$ or $\mu = \eta - 1$, then $|\langle M_k \rangle_\alpha| = \eta$.*

*(iii). If $\eta \geq 3$ and $\mu = \eta - i$ such that $2 \leq i \leq \eta - 1$, then $|\langle M_k \rangle_\alpha| = \eta + 2^{i-1} - 1$.*

**Corollary 3.4.** *Let $m \equiv 1 \pmod 4$ or $m \equiv 2 \pmod 4$ and let $\eta \in N$ . Then the orders of the semigroups $\langle M_3 \rangle_{2^\eta}$ are as follows:*

*i. If $m \equiv 1 \pmod 4$, then*

$$|\langle M_3 \rangle_{2^\eta}| = \begin{cases} 2 & \text{for } \eta = 1, \\ 4 & \text{for } \eta = 2, \\ 2^{\eta-2} + \eta & \text{for } \eta \geq 3. \end{cases}$$

*ii. If $m \equiv 2 \pmod 4$, then*

$$|\langle M_3 \rangle_{2^\eta}| = \begin{cases} 1 & \text{for } \eta = 1, \\ 2 & \text{for } \eta = 2, \\ 2^{\eta-2} + \eta - 1 & \text{for } \eta \geq 3. \end{cases}$$

**Corollary 3.5.** *Let $\eta \in N$. Then the orders of the semigroups $\langle M_4 \rangle_{3^\eta}$, $\langle M_5 \rangle_{2^\eta}$, $\langle M_5 \rangle_{7^\eta}$ and $\langle M_6 \rangle_{19^\eta}$ are as follows:*

$$|\langle M_4 \rangle_{3^\eta}| = \begin{cases} 2 \cdot 3^{\eta-1} + \eta - 1 & \text{for } \eta = 1, \\ 2 \cdot 3^{\eta-1} + \eta - 2 & \text{for } \eta \geq 2, \end{cases}$$

$$|\langle M_5 \rangle_{2^\eta}| = \begin{cases} 1 & \text{for } \eta = 1, \\ 2^{\eta-1} + \eta - 1 & \text{for } \eta \geq 2, \end{cases}$$

$$|\langle M_5 \rangle_{7^\eta}| = 48 \cdot 7^{\eta-1} + \eta - 1$$

*and*

$$|\langle M_6 \rangle_{19^\eta}| = 20 \cdot 19^{\eta-1} + \eta - 1.$$

**Corollary 3.6.** *Let $\eta \in N$. Then the orders of the semigroups $\langle M_k^* \rangle_{2^\eta}$ for $k = 1, 2, 3$ are as follows:*

*(i). If $m \equiv 0 \pmod 4$, then*

$$|\langle M_k^* \rangle_{2^\eta}| = \begin{cases} 3 & \text{for } \eta = 1, \\ 7 & \text{for } \eta = 2, \\ 2^{\eta-1} + 2^{\eta-2} + \eta - 1 & \text{for } \eta \geq 3. \end{cases}$$

*(ii). If $m \equiv 2 \pmod 4$, then $\left|\langle M_k^* \rangle_{2^\eta}\right| = 2^{\eta-1} + 2^{\eta-2} + \eta - 1$.*

*(iii). If $m$ is odd, then* $\left|\langle M_k^* \rangle_{2^\eta}\right| = 3\eta + 1$.

**Corollary 3.7.** *Let $\eta \in N$. Then the orders of the semigroups $\langle M_4^* \rangle_{3^\eta}$ and $\langle M_5^* \rangle_{2^\eta}$ are as follows:*

$$|\langle M_4^* \rangle_{3^\eta}| = 26 \cdot 3^{\eta-1} + \eta - 1$$

*and*

$$|\langle M_5^* \rangle_{2^\eta}| = 4\eta.$$

By an inductive argument for $n \geq 1$, we obtain

$$(M_1)^n = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}, \ (M_2)^n = \begin{bmatrix} x_4 & x_2 & x_5 \\ x_2 & x_1 & x_3 \\ x_7 & x_6 & x_8 \end{bmatrix}$$

and

$$\begin{aligned} (M_1^*)^n &= \begin{bmatrix} y_1 & y_2 & y_2 & y_3 \\ y_2 & y_4 & y_5 & y_6 \\ y_2 & y_5 & y_4 & y_6 \\ y_3 & y_6 & y_6 & y_7 \end{bmatrix}, \ (M_2^*)^n = \begin{bmatrix} y_4 & y_2 & y_5 & y_6 \\ y_2 & y_1 & y_2 & y_3 \\ y_5 & y_2 & y_4 & y_6 \\ y_6 & y_3 & y_6 & y_7 \end{bmatrix}, \\ (M_3^*)^n &= \begin{bmatrix} y_4 & y_5 & y_2 & y_6 \\ y_5 & y_4 & y_2 & y_6 \\ y_2 & y_2 & y_1 & y_3 \\ y_6 & y_6 & y_3 & y_7 \end{bmatrix}, \end{aligned}$$

where $x_i, y_j \in N$ such that $i = 1, \ldots, 8$ and $j = 1, \ldots, 7$. Thus, we have the following results

$$a_{3n+1}^1 = a_{3n+2}^2, \, a_{3n+2}^1 = a_{3n+1}^2, \, a_{3n+3}^1 = a_{3n+3}^2$$

and

$$\begin{aligned} b_{4n+1}^1 &= b_{4n+2}^2 = b_{4n+3}^3, \, b_{4n+2}^1 = b_{4n+1}^2 = b_{4n+1}^3, \, b_{4n+4}^1 = b_{4n+4}^2 = b_{4n+4}^3, \\ b_{4n+2}^1 &= b_{4n+3}^1, \, b_{4n+1}^2 = b_{4n+3}^2, \, b_{4n+1}^3 = b_{4n+2}^3 \end{aligned}$$

and hence

$$|\langle M_1 \rangle_\alpha| = |\langle M_2 \rangle_\alpha|, \ |\ \langle M_1^* \rangle_\alpha| = |\ \langle M_2^* \rangle_\alpha| = |\ \langle M_2^* \rangle_\alpha|$$

for every positive integer $\alpha$.

## 4. The Polyhedral Sequences in Groups

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period $k$ if the first $k$ elements in the sequence form a repeating subsequence.

Reducing 3-step and 4-step polyhedral sequences of the first, second, third, fourth, fifth and sixth kind by a modulus $\alpha$, then we get the repeating sequences, respectively denoted by

$$\left\{a_n^k(\alpha)\right\} = \left\{a_1^k(\alpha), a_2^k(\alpha), \ldots, a_i^k(\alpha), \ldots\right\}$$

and

$$\left\{b_n^k(\alpha)\right\} = \left\{b_1^k(\alpha), b_2^k(\alpha), \ldots, b_i^k(\alpha), \ldots\right\},$$

where $a_i^k(\alpha) = a_i^k \,(\mathrm{mod}\,\alpha)$, $b_i^k(\alpha) = b_i^k \,(\mathrm{mod}\,\alpha)$ and $k = 1, \ldots, 6$. The recurrence relations in the sequences $\left\{a_n^k(\alpha)\right\}$, $\left\{b_n^k(\alpha)\right\}$ and $\left\{a_n^k\right\}$, $\left\{b_n^k\right\}$ are the same, respectively.

**Theorem 4.1.** *For $k = 1, \ldots, 6$, the sequences $\left\{a_n^k(\alpha)\right\}$, $\left\{b_n^k(\alpha)\right\}$ are periodic.*

*Proof.* Let us consider the 4-step polyhedral sequence of the first kind $\left\{b_n^1(\alpha)\right\}$ as an example. Let $X = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid 0 \leq x_i \leq \alpha - 1\}$. Since there are $\alpha^7$ distinct 7-tuples of elements of $Z_\alpha$, at least one of the 7-tuples appears twice in the sequence $\left\{b_n^1(\alpha)\right\}$. Therefore, the subsequence following this 7-tuple repeats; that is the sequence is periodic.

There are similar proofs for the other sequences which are defined as the above. □

We next denote the periods of the sequences $\left\{a_n^k(\alpha)\right\}$ and $\left\{b_n^k(\alpha)\right\}$ by $l_{a^k}(\alpha)$ and $l_{b^k}(\alpha)$, respectively.

**Example 4.1.** *For $m = 2$, the sequence $\left\{b_n^1(3)\right\}$ is*

$$\{0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, \ldots\}$$

*and thus $l_{b^1}(3) = 8$.*

**Theorem 4.2.** *Let $\alpha$ be an positive integer and let $\left\{x_n^k(\alpha)\right\}$ be any of the sequences of $\left\{a_n^k(\alpha)\right\}$, $\left\{b_n^k(\alpha)\right\}$ for $k = 1, \ldots, 6$. If $\alpha$ has the prime factorization $\alpha = \prod_{j=1}^{t} p_j^{e_j}$, $(t \geq 1)$ and $(\alpha, \det M) = 1$ where $M$ is generating matrix of the sequence that is, $M = M_k$ or $M = M_k^*$, then*

$$l_{x^k}(\alpha) = lcm\left[l_{x^k}(p_1^{e_1}), l_{x^k}(p_2^{e_2}), \ldots, l_{x^k}(p_t^{e_t})\right].$$

*Proof.* Let us consider the 3-step polyhedral sequence of the fourth kind $\left\{a_n^4(\alpha)\right\}$ as an example. Since $l_{a^4}\left(p_j^{e_j}\right)$ is the length of the period of the sequence $\left\{a_n^k\left(p_j^{e_j}\right)\right\}$, this sequence repeats only after blocks of length $u \cdot l_{a^4}\left(p_j^{e_j}\right)$, $(u \in N)$. Since also $l_{a^4}(\alpha)$ is the length of the period of $\left\{a_n^k(\alpha)\right\}$, the sequence $\left\{a_n^k\left(p_j^{e_j}\right)\right\}$ repeats after $l_{a^4}(\alpha)$ terms for all values $j$. Thus, $l_{a^4}(\alpha)$ is of the form $u \cdot l_{a^4}\left(p_j^{e_j}\right)$ for all values $j$, and since any such number gives a period of $l_{a^4}(\alpha)$, we find that $l_{a^4}(\alpha) = lcm\left[l_{a^4}(p_1^{e_1}), l_{a^4}(p_2^{e_2}), \ldots, l_{a^4}(p_t^{e_t})\right]$.

There are similar proofs for the other sequences which are defined as the above. □

Since

$$(M_k)^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{3n+1}^k \\ a_{3n+2}^k \\ a_{3n+3}^k \end{bmatrix}$$

and

$$(M_k^*)^n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_{4n+1}^k \\ b_{4n+2}^k \\ b_{4n+3}^k \\ b_{4n+4}^k \end{bmatrix},$$

it is clear that $l_{a^k}(\alpha) = 3 \cdot |\langle M_k \rangle_\alpha|$ and $l_{b^k}(\alpha) = 4 \cdot \left|\langle M_k^* \rangle_\alpha\right|$ when $(\det M, \alpha) = 1$ where $M = M_k$ or $M = M_k^*$ for $k = 1, \ldots, 6$.

We next redefine the sequences $\left\{a_n^k\right\}$ and $\left\{b_n^k\right\}$ by means of the elements of the groups which have two or three generators.

**Definition 4.1.** *Let $G$ be a 2-generator group. For a generating pair $(x, y)$, we define the polyhedral 3-orbits of the first, second, third, fourth, fifth and sixth kind by:*

$$s_n^u = \begin{cases} \left(s_{n-3}^u\right)^{\alpha_1} s_{n-1}^u & n \equiv 1 \,(\operatorname{mod} 3), \\ \left(s_{n-3}^u\right)^{\alpha_2} s_{n-2}^u & n \equiv 2 \,(\operatorname{mod} 3), \\ \left(s_{n-5}^u\right)^{\alpha_3} \left(s_{n-4}^u\right)^{\alpha_3} s_{n-3}^u & n \equiv 0 \,(\operatorname{mod} 3), \end{cases},$$

$$(u = 1, 2, 3,\ \alpha_u = m \text{ and } \alpha_i = 2 \text{ if } i \neq u)$$

$$s_n^v = \begin{cases} \left(s_{n-3}^v\right)^2 s_{n-1}^v & n \equiv 1 \,(\operatorname{mod} 3), \\ \left(s_{n-3}^v\right)^3 s_{n-2}^v & n \equiv 2 \,(\operatorname{mod} 3), \\ \left(s_{n-5}^v\right)^{v-1} \left(s_{n-4}^v\right)^{v-1} s_{n-3}^v & n \equiv 0 \,(\operatorname{mod} 3), \end{cases} \quad (v = 4, 5, 6)$$

for $n \geq 4$, with initial conditions $s_1^k = x$, $s_2^k = y$, $s_3^k = y$, $(k = 1, \ldots, 6)$.

For a generating pair $(x, y)$, the polyhedral 3-orbits of the first, second, third, fourth, fifth and sixth kind are denoted by $O_{(x,y)}^{3,1}(G)$, $O_{(x,y)}^{3,2}(G)$, $O_{(x,y)}^{3,3}(G)$, $O_{(x,y)}^{3,4}(G)$, $O_{(x,y)}^{3,5}(G)$ and $O_{(x,y)}^{3,6}(G)$, respectively.

**Definition 4.2.** *Let $G$ be a 3-generator group. For a generating triple $(x, y, z)$, we define the polyhedral 4-orbits of the first, second, third, fourth, fifth and sixth kind by:*

$$r_n^u = \begin{cases} \left(r_{n-4}^u\right)^{\alpha_1} r_{n-1}^u & n \equiv 1 \,(\operatorname{mod} 4), \\ \left(r_{n-4}^u\right)^{\alpha_2} r_{n-2}^u & n \equiv 2 \,(\operatorname{mod} 4), \\ \left(r_{n-4}^u\right)^{\alpha_3} r_{n-3}^u & n \equiv 3 \,(\operatorname{mod} 4), \\ r_{n-7}^u r_{n-6}^u r_{n-5}^u r_{n-4}^u & n \equiv 0 \,(\operatorname{mod} 4), \end{cases} \quad (u = 1, 2, 3,\ \alpha_u = m \textit{ and } \alpha_i = 2 \textit{ if } i \neq u),$$

$$r_n^v = \begin{cases} \left(r_{n-4}^v\right)^2 r_{n-1}^v & n \equiv 1 \,(\operatorname{mod} 4), \\ \left(r_{n-4}^v\right)^3 r_{n-2}^v & n \equiv 2 \,(\operatorname{mod} 4), \\ \left(r_{n-4}^v\right)^{(v-1)} r_{n-3}^v & n \equiv 3 \,(\operatorname{mod} 4), \\ r_{n-7}^v r_{n-6}^v r_{n-5}^v r_{n-4}^v & n \equiv 0 \,(\operatorname{mod} 4), \end{cases} \quad (v = 4, 5, 6)$$

*for $n \geq 5$, with initial conditions $r_1^k = x$, $r_2^k = y$, $r_3^k = z$, $r_4^k = z$, $(k = 1, \ldots, 6)$.*

For a generating triple $(x, y, z)$, the polyhedral 4-orbits of the first, second, third, fourth, fifth and sixth kind are denoted by $O^{4,1}_{(x,y,z)}(G)$, $O^{4,2}_{(x,y,z)}(G)$, $O^{4,3}_{(x,y,z)}(G)$, $O^{4,4}_{(x,y,z)}(G)$, $O^{4,5}_{(x,y,z)}(G)$ and $O^{4,6}_{(x,y,z)}(G)$, respectively.

**Theorem 4.3.** *The polyhedral* 3*-orbits and* 4*-orbits of the first, second, third, fourth, fifth and sixth kind of a finite group $G$ are periodic.*

*Proof.* Let us consider the polyhedral 3-orbit of the first kind $O^{3,1}_{(x,y)}(G)$ as an example. Suppose that $n$ is the order of $G$. Since there are $n^5$ distinct 5-tuples of elements of $G$, at least one of the 5-tuples appears twice in the sequence $O^{3,1}_{(x,y)}(G)$. Therefore, the subsequence following this 5-tuple repeats. Because of the repetition, the sequence is periodic. □

We denote the lengths of the periods of the orbits $O^{3,k}_{(x,y)}(G)$ and $O^{4,k}_{(x,y,z)}(G)$ by $LO^{3,k}_{(x,y)}(G)$ and $LO^{4,k}_{(x,y,z)}(G)$ for $k = 1, \ldots, 6$, respectively.

We will now address the lengths of the periods of the polyhedral 3-orbits and 4-orbits of the first, second, third, fourth, fifth and sixth kind of finite polyhedral groups as applications of the results obtained.

**Theorem 4.4.** *The orbit $O^{3,1}_{(x,y)}((m,2,2))$ is a simply periodic sequence and $LO^{3,1}_{(x,y)}((m,2,2)) = 6i$ where $i$ is the least positive integer such that $(-2)^i \equiv 1 \pmod{m}$ and*
$\left[(-2)^i + (-2)^{i-1} + \cdots + (-2)^3\right] + 2 \equiv 0 \pmod{m}$.

*Proof.* We first note that the polyhedral group $(m, 2, 2)$ of order $2m$ is presented in the 2-generator case by

$$\left\langle x, y \mid x^m = y^2 = (xy)^2 = e \right\rangle .$$

The sequence $O^{3,1}_{(x,y)}((m,2,2))$ is

$$x, y, y, y, y, x^2y, \ldots .$$

Using the above, the sequence becomes:

$$\begin{aligned}
s^1_1 &= x,\ s^1_2 = y,\ s^1_3 = y,\ s^1_4 = y,\ s^1_5 = y,\ s^1_6 = x^2y, \ldots, \\
s^1_{6i+1} &= x^{(-2)^i},\ s^1_{6i+2} = s^1_{6i+3} = s^1_{6i+4} = s^1_{6i+5} = x^{-\left[(-2)^i + (-2)^{i-1} + \cdots + (-2)^3\right] - 2} y, \\
s^1_{6i+6} &= x^{-\left[(-2)^{i+1} + (-2)^i + \cdots + (-2)^3\right] - 2} y, \ldots .
\end{aligned}$$

So we need the smallest positive integer $i$ such that

$$(-2)^i = um + 1 \text{ and } \left[(-2)^i + (-2)^{i-1} + \cdots + (-2)^3\right] + 2 = vm \text{ for } u, v \in N.$$

Thus the proof is complete. □

**Theorem 4.5.**

$$LO_{(x,y)}^{3,2}\left((2,m,2)\right)=l_{a^2}\left(m\right), LO_{(x,y)}^{3,3}\left((2,2,m)\right)=3, LO_{(x,y)}^{3,4}\left((2,3,3)\right)=18,$$
$$LO_{(x,y)}^{3,5}\left((2,3,4)\right)=9, LO_{(x,y)}^{3,6}\left((2,3,5)\right)=21$$

*and*

$$LO_{(x,y,z)}^{4,1}\left((m,2,2)\right)=LO_{(x,y,z)}^{4,2}\left((2,m,2)\right)=\begin{cases} 4 & \text{if } m \text{ is odd,} \\ 12 & \text{if } m \text{ is even,} \end{cases}$$
$$LO_{(x,y,z)}^{4,3}\left((2,2,m)\right)=l_{b^3}\left(m\right), LO_{(x,y,z)}^{4,4}\left((2,3,3)\right)=104,$$
$$LO_{(x,y,z)}^{4,5}\left((2,3,4)\right)=4, LO_{(x,y,z)}^{4,6}\left((2,3,5)\right)=248.$$

*Proof.* Let us consider the polyhedral 4-orbit of the third kind of the polyhedral group $(2,2,m)$, $O_{(x,y,z)}^{4,3}\left((2,2,m)\right)$ as an example. The sequence $O_{(x,y,z)}^{4,3}\left((2,2,m)\right)$ is

$$x, y, z, z, z, z, z, z, z, z^3, z^3, z, z^4, z^{10}, z^{10}, z^4, z^{11}, \ldots.$$

Using the above, the sequence becomes:

$$\begin{array}{rcl} r_5^3 & = & z=z^{b_5^3},\ r_6^3=z=z^{b_6^3},\ r_7^3=z=z^{b_7^3},\ r_8^3=z=z^{b_8^3},\ldots, \\ r_{4i+1}^3 & = & z^{b_{4i+1}^3},\ r_{4i+2}^3=z^{b_{4i+2}^3},\ r_{4i+3}^3=z^{b_{4i+3}^3},\ r_{4i+4}^3=z^{b_{4i+4}^3},\ldots. \end{array}$$

Since the order of $z$ is $m$, it is easy to see that the length of the period of the orbit $O_{(x,y,z)}^{4,3}\left((2,2,m)\right)$ is $l_{b^3}\left(m\right)$.

There are similar proofs for the other orbits. □

**Acknowledgement.** This Project was supported by the Commission for the Scientific Research Projects of Kafkas University. The Project number. 2015-FM-45.

## References


[1] Aydın, H. and Smith, G. C., Finite p-quotients of some cyclically presented groups, *J. Lond. Math. Soc.* 49 (1994), 83-92.

[2] Bozkurt, D. and Tam, T-Y., Determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers, *Appl. Math. Comput.* (2012), 219(2), 544-551.

[3] Campbell, C. M. and Campbell, P.P., The Fibonacci lengths of binary polyhedral groups and related groups, *Congr. Numer.* 194 (2009), 95-102.

[4] Coxeter, H. S. M. and Moser, W. O. J., Generators and relations for discrete groups, 3rd edition, Springer, Berlin, 1972.

[5] Deveci, O., On the Fibonacci-circulant p-sequences, *Util. Math.* in press.

[6] Deveci, O. and Akuzum,Y., The recurrence sequences via Hurwitz matrices, *Sci. Ann. "Al. I. Cuza" Univ. Iasi, in press.*

[7] Deveci, O. and Karaduman, E., The cyclic groups via the Pascal matrices and the generalized Pascal matrices, *Linear Algebra Appl.* 437 (2012), 2538-2545.

[8] Doostie, H. and Campbell, C. M., Fibonacci length of automorphism groups involving tribonacci numbers, *Vietnam J. Math.* (2000), 28(1), 57-65.

[9] Falcon, S. and Plaza, A., k-Fibonacci sequences modulo m, *Chaos Solitons Fractals* (2009), 41(1), 497-504.
[10] Frey, D. D. and Sellers, J. A., Jacobsthal numbers and alternating sign matrices, *J. Integer Seq.* 3 (2000), Article 00.2.3.
[11] Gogin, N. D. and Myllari, A. A., The Fibonacci-Padovan sequence and MacWilliams transform matrices, *Program. Comput. Softw., published in Programmirovanie* (2007), 33(2), 74-79.
[12] Johnson, D .L., Topics in the theory of group presentations, London Math. Soc. Lecture Notes, Cambridge University Press, 1980.
[13] Kalman, D., Generalized Fibonacci numbers by matrix methods, *Fibonacci Quart.* (1982), 20(1), 73-76.
[14] Karaduman, E. and Deveci, O., k-nacci sequences in finite triangle groups, *Disc. Dyn. Nat. Soc.* (2009), 453750-5-453750-10.
[15] Kilic, E., The generalized Pell (p,i)-numbers and their Binet formulas, combinatorial representations, sums, *Chaos, Solitons Fractals* (2009), 40(4), 2047-2063.
[16] Knox, S. W., Fibonacci sequences in finite groups, *Fibonacci Quart.* (1992), 30(2), 116-120.
[17] Lu, K. and Wang, J., k-step Fibonacci sequence modulo m, *Util. Math.* 71 (2006), 169-178.
[18] Ozkan, E., On truncated Fibonacci sequences, *Indian J. Pure Appl. Math.* (2007), 38(4), 241-251.
[19] Spinadel, V. W., The metallic means family and forbidden symmetries, *Int. Math. J.* (2002), 2(3), 279-288.
[20] Stakhov, A. P. and Rozin, B., Theory of Binet formulas for Fibonacci and Lucas p-numbers, *Chaos Solitons Fractals* (2006), 27(5), 1162-1177.
[21] Tasci, D. and Firengiz, M .C., Incomplete Fibonacci and Lucas p-numbers, *Math. Comput. Modelling* 52 (2010), 1763-1770.
[22] Tuglu, N., Kocer, E. G. and Stakhov, A. P., Bivariate Fibonacci like p-polynomials, *Appl. Math. Comput.* (2011), 217(24), 10239-10246.
[23] Wall, D. D., Fibonacci series modulo m , *Amer. Math. Monthly* 67 (1960), 525-532.
[24] Wilcox, H.J., Fibonacci sequences of period n in groups, *Fibonacci Quart.* (1986), 24(4), 356-361.

*Current address*: Ömür Deveci: Faculty of Science and Letters, Kafkas University 36100, Turkey

*E-mail address*: odeveci36@hotmail.com

ORCID: http://orcid.org/0000-0001-5870-5298

*Current address*: Yeşim Aküzüm: Faculty of Science and Letters, Kafkas University 36100, Turkey

*E-mail address*: yesim_036@hotmail.com

ORCID: http://orcid.org/0000-0001-7168-8429

*Current address*: Colin M. Campbell: School of Mathematics and Statistics, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

*E-mail address*: cmc@st-andrews.ac.uk