Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (5) (2020), 1695 – 1705
DOI : 10.15672/hujms.557072

Research Article

Fekete-Szegö problem for generalized bi-subordinate functions of complex order

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Abstract

In this paper, we obtain the Fekete-Szegö inequality for the generalized bi-subordinate functions of complex order. The various results, which are presented in this paper, would generalize those in related works of several earlier authors.

Mathematics Subject Classification (2010). 30C45, 30C80

Keywords. analytic functions, starlike functions, convex functions, Ma-Minda starlike functions, Ma-Minda convex functions, subordination, Fekete-Szegö inequality

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the class of functions f that are analytic, univalent in $\mathbb{D}$ and are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

The Koebe one-quarter theorem assures that the image of unit disk $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \ (z \in \mathbb{D}) \text{ and } f(f^{-1}(w)) = w, \quad (|w| < r_0, \ r_0 \geq 1/4).$$

Furthermore, the Taylor-Maclaurin series of f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if f is univalent and f^{-1} has univalent analytic continuation, which we denote by g, to the unit disk $\mathbb{D}$. Let σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. Coefficient problem for bi-univalent functions were recently investigated by several authors [1,4–8,15–17,19,20]. A function $f \in \mathcal{A}$ is said to be subordinate to a function $h \in \mathcal{A}$, denoted by $f \prec h$, if there exists an analytic function $w \in \mathcal{B}_0$, where $\mathcal{B}_0 := \{w : w(0) = 0, \ |w(z)| < 1, \ z \in \mathbb{D}\}$ such that $f(z) = h(w(z))$. We let $\mathcal{S}^*$ consist of starlike functions $f \in \mathcal{A}$, that is, $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ in $\mathbb{D}$ and $\mathcal{C}$ consist of convex functions $f \in \mathcal{A}$, that is, $1 + \mathrm{Re}\{zf''(z)/f'(z)\} > 0$ in $\mathbb{D}$. Ma and

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Received: 22.04.2019; Accepted: 13.12.2019

Minda [12] unified various subclasses of starlike and convex functions for which either of the quantity $zf'(z)/f(z)$ or $1+ zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\mathbb{D}$ and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) \prec \varphi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $1+ zf''(z)/f'(z) \prec \varphi(z)$. Extensions of the above two classes (see [14]) are

$$\mathcal{S}^*(\gamma;\varphi) \equiv \left\{ f \in \mathcal{A} : 1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z), \quad \gamma \in \mathbb{C}\backslash \{0\} \right\}$$

and

$$C(\gamma;\varphi) \equiv \left\{ f \in \mathcal{A} : 1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right) \prec \varphi(z), \quad \gamma \in \mathbb{C}\backslash \{0\} \right\}.$$

In literature, the functions belonging to these classes are called Ma-Minda starlike and convex of complex order γ $(\gamma \in \mathbb{C}\diagdown\{0\})$, respectively. A function f is bi-starlike of Ma-Minda type of complex order γ $(\gamma \in \mathbb{C}\diagdown\{0\})$ and bi-convex of Ma-Minda type of complex order γ $(\gamma \in \mathbb{C}\diagdown\{0\})$ if both f and g are ,respectively, Ma-Minda starlike and convex of complex order γ $(\gamma \in \mathbb{C}\diagdown\{0\})$. The classes consisting of bi-starlike of Ma-Minda type of complex order γ $(\gamma \in \mathbb{C}\diagdown\{0\})$ and bi-convex of Ma-Minda type of complex order γ $(\gamma \in \mathbb{C}\diagdown\{0\})$ are denoted by $\mathcal{S}^*_\sigma(\gamma;\varphi)$ and $\mathcal{C}_\sigma(\gamma;\varphi)$, respectively. As a special case $\gamma = 1$ the classes $\mathcal{S}^*_\sigma(\gamma;\varphi)$ and $\mathcal{C}_\sigma(\gamma;\varphi)$ reduce to bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type functions are denoted by $\mathcal{S}^*_\sigma(\varphi)$ and $\mathcal{C}_\sigma(\varphi)$, respectively.

In this paper, we consider more general class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$ for $0 \le \lambda \le 1$, $\gamma \in \mathbb{C}\backslash \{0\}$ which was investigated by Deniz [5] wherein he obtained the bounds for a_2 and a_3. This motivated us to study the Fekete-Szegö inequality to the class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$. Recently, some authors have investigated the Fekete-Szegö problem for various subclasses of σ (see [3,9,13,21,22]).

2. Coefficient estimates

Throughout this paper φ denotes an analytic univalent function in $\mathbb{D}$ with positive real part and normalized by $\varphi(0) = 1$, $\varphi'(0) > 0$. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \quad (B_1 > 0)\,. \tag{2.1}$$

Definition 2.1. For $0 \le \lambda \le 1$ and $\gamma \in \mathbb{C}\backslash \{0\}$, the class $\mathcal{S}(\lambda,\gamma;\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right) \prec \varphi(z) \quad (z \in \mathbb{D}).$$

The class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$ consists of functions $f \in \sigma$ such that $f, g \in \mathcal{S}(\lambda,\gamma;\varphi)$ where g is the analytic continuation of f^{-1} to the unit disk $\mathbb{D}$.

The class $\mathcal{S}(\lambda,\gamma;\varphi)$ was introduced by [18]. Motivated by this class the second author [5] defined and studied the class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$, which is called the class of generalized bi-subordinate functions of complex order γ and type λ. As special cases of the class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$, we have $\mathcal{S}_\sigma(0,\gamma;\varphi) \equiv \mathcal{S}^*_\sigma(\gamma;\varphi)$ and $\mathcal{S}_\sigma(1,\gamma;\varphi) \equiv \mathcal{C}_\sigma(\gamma;\varphi)$.

The class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$ includes many earlier classes, which are mentioned below:
$\mathcal{S}_\sigma(0,1;\varphi) \equiv \mathcal{S}^*_\sigma(\varphi)$ and $\mathcal{S}_\sigma(1,1;\varphi) \equiv \mathcal{C}_\sigma(\varphi)$, are classes of Ma-Minda bi-starlike and Ma-Minda bi-convex functions, respectively, introduced and studied in [11].

$\mathcal{S}_\sigma((0,1;(1+Az)/(1+Bz)) \equiv \mathcal{S}_\sigma[A,B]$ and $\mathcal{S}_\sigma(1,1;(1+Az)/(1+Bz)) \equiv \mathcal{C}_\sigma[A,B]$ $(-1 \le B < A \le 1)$ are, respectively, the classes of Janowski bi-starlike and bi-convex functions. Additionally, for $0 \le \beta < 1$, $\mathcal{S}_\sigma[1-2\beta,1] \equiv \mathcal{S}_\sigma(\beta)$ and $\mathcal{C}_\sigma[1-2\beta,1] \equiv \mathcal{C}_\sigma(\beta)$ are, respectively, the classes of bi-starlike and bi-convex functions of order β introduced and studied in [2].

For $0 < \beta \leq 1$, $\mathcal{S}_\sigma\left(0,1;\left(\frac{1+z}{1-z}\right)^\beta\right) \equiv \mathcal{SS}^*_\sigma(\beta)$ and $\mathcal{S}_\sigma\left(1,1;\left(\frac{1+z}{1-z}\right)^\beta\right) \equiv \mathcal{SC}^*_\sigma(\beta)$ are, respectively, classes of strongly bi-starlike and strongly bi-convex functions of order β introduced and studied in [2].

For $\gamma \in \mathbb{C}\diagdown\{0\}$, $\mathcal{S}_\sigma(0,\gamma;(1+z)/(1-z)) \equiv \mathcal{S}^*_\sigma[\gamma]$ and $\mathcal{S}_\sigma(1,\gamma;(1+z)/(1-z)) \equiv \mathcal{C}_\sigma[\gamma]$ are classes of bi-starlike and bi-convex functions of complex order, respectively.

To prove our next theorems, we shall need the following well-known lemma (see [10]).

Lemma 2.2 ([10]). *Let the function $w \in \mathcal{B}_0$ be given by*

$$w(z) = c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{D}),$$

then for by every complex number s,

$$\left|c_2 - s c_1^2\right| \leq 1 + (|s| - 1)\,|c_1|^2 .$$

In the following theorem, we consider functional $\left|a_3 - \mu a_2^2\right|$ for γ nonzero complex number and $\mu \in \mathbb{C}$.

Theorem 2.3. *Let the function f given by (1.1) be in the $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$. For $\gamma \in \mathbb{C}\diagdown\{0\}$ and $\mu \in \mathbb{C}$, we have*

$$|a_2| \leq \frac{|\gamma|\,B_1}{1+\lambda}, \tag{2.2}$$

$$|a_3| \leq \frac{|\gamma|\,|B_1|}{4\,(1+2\lambda)} \max\{2, (|s| + |t|)\} \tag{2.3}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_1|\gamma|}{2(1+2\lambda)} & \text{if } \mathcal{L} \leq 2 \\ \frac{B_1|\gamma|}{4(1+2\lambda)}\mathcal{L} & \text{if } \mathcal{L} > 2 \end{cases} \tag{2.4}$$

where $s = \frac{B_2}{B_1} - \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}$, $t = \frac{B_2}{B_1}$ *and* $\mathcal{L} = \left|\frac{B_2}{B_1} + (1-\mu)\,\frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}\right| + \left|\frac{B_2}{B_1}\right|$.

Proof. Since $f \in \mathcal{S}_\sigma(\lambda,\gamma;\varphi)$, there exists two analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = 0 = v(0)$, such that

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right) = \varphi(u(z)) \qquad (z \in \mathbb{D}) \tag{2.5}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda w g'(w)} - 1\right) = \varphi(v(w)). \tag{2.6}$$

Define the functions u and v by

$$u(z) = c_1 z + c_2 z^2 + \cdots \text{ and } v(w) = d_1 w + d_2 w^2 + \cdots . \tag{2.7}$$

Using (2.1) with (2.7), it is evident that

$$\varphi(u(z)) = 1 + (B_1 c_1)z + (B_1 c_2 + B_2 c_1^2)z^2 + \cdots \tag{2.8}$$

and

$$\varphi(v(w)) = 1 + (B_1 d_1)w + (B_1 d_2 + B_2 d_1^2)w^2 + \cdots . \tag{2.9}$$

Also, using (1.1), we get

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right) = 1 + \frac{(1+\lambda)\,a_2}{\gamma}z + \left[\frac{2\,(1+2\lambda)\,a_3 - (1+\lambda)^2\,a_2^2}{\gamma}\right]z^2 + \cdots \tag{2.10}$$

and using (1.2), we get

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)+\lambda w^2g''(w)}{(1-\lambda)g(w)+\lambda wg'(w)}-1\right)$$
$$=1-\frac{(1+\lambda)\,a_2}{\gamma}w\left[\frac{-2\,(1+2\lambda)\,a_3+\left(3+6\lambda-\lambda^2\right)a_2^2}{\gamma}\right]w^2+\cdots. \tag{2.11}$$

Equating coefficients of right sides of equations (2.8) with (2.10) and (2.9) with (2.11) yield

$$\frac{(1+\lambda)\,a_2}{\gamma}=B_1c_1,\quad \frac{2\,(1+2\lambda)\,a_3-(1+\lambda)^2\,a_2^2}{\gamma}=B_1c_2+B_2c_1^2 \tag{2.12}$$

and

$$\frac{-\,(1+\lambda)\,a_2}{\gamma}=B_1d_1,\quad \frac{-2\,(1+2\lambda)\,a_3+\left(3+6\lambda-\lambda^2\right)a_2^2}{\gamma}=B_1d_2+B_2d_1^2 \tag{2.13}$$

so that, on account of (2.12) and (2.13)

$$c_1 = -d_1, \tag{2.14}$$
$$a_2 = \frac{\gamma B_1}{1+\lambda}c_1 \tag{2.15}$$

and

$$a_3=a_2^2+\frac{\gamma}{4\,(1+2\lambda)}\left[B_1c_2+B_2c_1^2-B_1d_2-B_2d_1^2\right]. \tag{2.16}$$

Taking into account (2.14), (2.15), (2.16) and the well known estimate $|c_1|\leq 1$ of the Schwarz lemma, we get

$$|a_2|=\left|\frac{\gamma B_1}{1+\lambda}c_1\right|\leq\frac{|\gamma|\,B_1}{1+\lambda} \tag{2.17}$$

and from Lemma 2.2,

$$\begin{aligned}
|a_3| &= \left|a_2^2+\frac{\gamma}{4\,(1+2\lambda)}\left[B_1c_2+B_2c_1^2-B_1d_2-B_2d_1^2\right]\right|\\
&= \left|\frac{\gamma^2B_1^2}{(1+\lambda)^2}c_1^2+\frac{\gamma}{4\,(1+2\lambda)}\left[\left(B_1c_2-B_2c_1^2\right)-\left(B_1d_2-B_2d_1^2\right)\right]\right|\\
&= \left|\frac{\gamma B_1}{4\,(1+2\lambda)}\left\{\left[c_2-\left(\frac{B_2}{B_1}-\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right)c_1^2\right]-\left[d_2-\frac{B_2}{B_1}d_1^2\right]\right\}\right|\\
&\leq \frac{|\gamma|\,B_1}{4\,(1+2\lambda)}\left\{\left|c_2-\left(\frac{B_2}{B_1}-\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right)c_1^2\right|+\left|d_2-\frac{B_2}{B_1}d_1^2\right|\right\}\\
&\leq \frac{|\gamma|\,B_1}{4\,(1+2\lambda)}\left\{1+(|s|-1)\left|c_1^2\right|+1+(|t|-1)\left|c_1^2\right|\right\}\\
&= \frac{|\gamma|\,B_1}{4\,(1+2\lambda)}\left\{2+(|s|+|t|-2)\left|c_1^2\right|\right\}.
\end{aligned}$$

Thus, using $|c_1|\leq 1$ we have the desired estimate for $|a_3|$:

$$|a_3|\leq\frac{|\gamma|\,|B_1|}{4\,(1+2\lambda)}\max\{2,(|s|+|t|)\},$$

where $s=\frac{B_2}{B_1}-\frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}$ and $t=\frac{B_2}{B_1}$.

To find an estimate for $\left|a_3-\mu a_2^2\right|$, we express $a_3-\mu a_2^2$ in terms of c_i and d_i. Using the equality (2.16), we have

$$a_3-\mu a_2^2=(1-\mu)\,a_2^2+\frac{\gamma}{4\,(1+2\lambda)}\left[B_1c_2+B_2c_1^2-B_1d_2-B_2d_1^2\right].$$

Therefore from Lemma 2.2, we obtain

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &= \left|(1-\mu)\,a_2^2+\frac{\gamma}{4\,(1+2\lambda)}\left[B_1c_2+B_2c_1^2-B_1d_2-B_2d_1^2\right]\right| \\ &= \left|\frac{\gamma B_1}{4\,(1+2\lambda)}\left\{\left[c_2-\left(\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right)c_1^2\right]-\left[d_2-\frac{B_2}{B_1}d_1^2\right]\right\}\right| \\ &\leq \frac{|\gamma|\,B_1}{4\,(1+2\lambda)}\left\{2+\left(\left|\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right|+\left|\frac{B_2}{B_1}\right|-2\right)\left|c_1^2\right|\right\}. \quad (2.18)\end{aligned}$$

As a result of this, from $|c_1|\leq 1$ we obtain

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{B_1|\gamma|}{2(1+2\lambda)} & \text{if } \mathcal{L}<2,\\ \frac{B_1|\gamma|}{4(1+2\lambda)}\mathcal{L} & \text{if } \mathcal{L}\geq 2,\end{cases}$$

where $\mathcal{L}=\left|\frac{B_2}{B_1}+(1-\mu)\,\frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}\right|+\left|\frac{B_2}{B_1}\right|$.
Thus the proof is completed. □

We next consider the cases γ and μ are real.

Theorem 2.4. *Let the function f given by (1.1) be in the $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$. For $\gamma>0$ and $\mu\in\mathbb{R}$, we have*

(1) *If $|B_2|\geq B_1$, then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{\gamma|B_2|}{2(1+2\lambda)}-(\mu-1)\,\frac{\gamma^2B_1^2}{(1+\lambda)^2} & \text{if } \mu\leq 1\\ \frac{\gamma|B_2|}{2(1+2\lambda)}+(\mu-1)\,\frac{\gamma^2B_1^2}{(1+\lambda)^2} & \text{if } \mu>1\end{cases}.$$

(2) *If $|B_2|<B_1$, then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{\gamma|B_2|}{2(1+2\lambda)}-(\mu-1)\,\frac{\gamma^2B_1^2}{(1+\lambda)^2} & \text{if } \mu\leq 1-\mathcal{F}\\ \frac{\gamma B_1}{2(1+2\lambda)} & \text{if } 1-\mathcal{F}<\mu<1+\mathcal{F}\\ \frac{\gamma|B_2|}{2(1+2\lambda)}+(\mu-1)\,\frac{\gamma^2B_1^2}{(1+\lambda)^2} & \text{if } \mu\geq 1+\mathcal{F}\end{cases}$$

where $\mathcal{F}=\frac{(1+\lambda)^2(B_1-|B_2|)}{2\gamma B_1^2(1+2\lambda)}$.

Proof. Using (2.18) and Lemma 2.2, we obtain

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &= \left|\frac{\gamma B_1}{4\,(1+2\lambda)}\left\{\left[c_2-\left(\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right)c_1^2\right]-\left[d_2-\frac{B_2}{B_1}d_1^2\right]\right\}\right| \\ &\leq \frac{\gamma B_1}{4\,(1+2\lambda)}\left\{2+\left(\left|\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right|+\left|\frac{B_2}{B_1}\right|-2\right)\left|c_1^2\right|\right\} \\ &\leq \frac{\gamma B_1}{2\,(1+2\lambda)}+\left\{\frac{\gamma\,(|B_2|-B_1)}{2\,(1+2\lambda)}+|\mu-1|\,\frac{\gamma^2B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right|. \quad (2.19)\end{aligned}$$

Now, the proof will be presented in two cases:

Firstly, we consider the case $|B_2| \geq B_1$.
If $\mu \leq 1$, then using (2.19) and $|c_1| \leq 1$, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\} \\ &= \frac{\gamma |B_2|}{2(1+2\lambda)} - (\mu - 1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}.\end{aligned}$$

If $\mu > 1$, then using (2.19) and $|c_1| \leq 1$, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\} \\ &= \frac{\gamma |B_2|}{2(1+2\lambda)} + (\mu - 1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}.\end{aligned}$$

Finally, we consider the case $|B_2| < B_1$. By using (2.19) and $|c_1| \leq 1$, we obtain the following results according to the cases of μ and $\mathcal{F}$.
For $\mu \leq 1 - \mathcal{F}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\} \\ &= \frac{\gamma |B_2|}{2(1+2\lambda)} - (\mu - 1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2},\end{aligned}$$

and for $1 - \mathcal{F} < \mu \leq 1$, we yield

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)}.\end{aligned}$$

Similarly for $1 < \mu < 1 + \mathcal{F}$, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)}.\end{aligned}$$

Finally for $\mu \geq 1 + \mathcal{F}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\} \\ &= \frac{\gamma |B_2|}{2(1+2\lambda)} + (\mu - 1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}.\end{aligned}$$

Thus the proof is completed. □

Finally, we consider the cases of γ nonzero complex number and $\mu \in \mathbb{R}$.

Theorem 2.5. *Let the function f given by (1.1) be in the $\mathcal{S}_\sigma(\lambda, \gamma; \varphi)$. For $\gamma \in \mathbb{C}\setminus\{0\}$ and $\mu \in \mathbb{R}$, we have*

(1) *If* $\frac{(1+|\sin\theta|)|B_2|}{2B_1} \geq 1$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re(k_1)\right) + \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} & \text{if } \mu \leq 1-\Re(k_1) \\ \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re(k_1)\right) & \text{if } \mu > 1-\Re(k_1) \end{cases}.$$

(2) *If* $\frac{(1+|\sin\theta|)|B_2|}{2B_1} < 1$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re(k_1)\right) + \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} & \text{if } \mu \leq 1-\Re(k_1)+\mathcal{N} \\ \frac{|\gamma|B_1}{2(1+2\lambda)} & \text{if } 1-\Re(k_1)+\mathcal{N}<\mu<1-\Re(k_1)-\mathcal{N} \\ \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re(k_1)\right) & \text{if } \mu \geq 1-\Re(k_1)-\mathcal{N} \end{cases}$$

where $k_1 = \frac{B_2(1+\lambda)^2 e^{i\theta}}{4B_1^2|\gamma|(1+2\lambda)}$, $|\gamma| = \gamma e^{i\theta}$ *and* $\mathcal{N} = \frac{(1+\lambda)^2[|B_2|(1+|\sin\theta|)-2B_1]}{4B_1^2|\gamma|(1+2\lambda)}$.

Proof. Let $f \in \mathcal{S}_\sigma(\lambda, \gamma; \varphi)$. By using (2.18) and Lemma 2.2, then we obtain

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{4(1+2\lambda)}\left\{2 + \left(\left|\frac{B_2}{B_1} - (1-\mu)\frac{4\gamma B_1(1+2\lambda)}{(1+\lambda)^2}\right| + \left|\frac{B_2}{B_1}\right| - 2\right)\left|c_1^2\right|\right\} \\ &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \\ &\quad \times \left[\left|(1-\mu) - \frac{B_2(1+\lambda)^2}{4B_1^2\gamma(1+2\lambda)}\right| + \frac{(|B_2| - 2B_1)(1+\lambda)^2}{4B_1^2|\gamma|(1+2\lambda)}\right]\left|c_1^2\right|. \end{aligned}$$

Taking $|\gamma| = \gamma e^{i\theta}$, $k_1 = \frac{B_2(1+\lambda)^2 e^{i\theta}}{4B_1^2|\gamma|(1+2\lambda)}$ and $l_1 = \frac{(|B_2|-2B_1)(1+\lambda)^2}{4B_1^2|\gamma|(1+2\lambda)}$, for B_1, $B_2 \in \mathbb{R}$ and $B_1 > 0$, we rewrite

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(|1-\mu-k_1| + l_1\right)\left|c_1^2\right| \qquad (2.20) \\ &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(|1-\mu-\Re(k_1) - i(\mathbf{Im}(k_1)| + l_1\right)\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(|1-\mu-\Re(k_1)| + \frac{|B_2|(1+\lambda)^2|\sin\theta|}{4B_1^2|\gamma|(1+2\lambda)} + l_1\right)\left|c_1^2\right| \\ &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}|1-\mu-\Re(k_1)| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right|. \end{aligned}$$

Firstly, we consider the case $\frac{(1+|\sin\theta|)|B_2|}{2B_1} \geq 1$.
Let $\mu \leq 1 - \Re(k_1)$. Then from (2.20) and $|c_1| \leq 1$, we obtain

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}|1-\mu-\Re(k_1)| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re(k_1)\right) + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)} \\ &= \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re(k_1)\right) + \frac{|\gamma|\,|B_2|(1+|\sin\theta|)}{4(1+2\lambda)}. \end{aligned}$$

Let $\mu > 1 - \Re(k_1)$. Then from (2.20) and $|c_1| \leq 1$, we yield

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}|1-\mu-\Re(k_1)| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(\mu+\Re(k_1)-1) + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)} \\ &= \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(1-\mu-\Re(k_1)).\end{aligned}$$

Finally, we want to consider the case with $\frac{(1+|\sin\theta|)|B_2|}{2B_1} < 1$. By using (2.20) and $|c_1| \leq 1$, we obtain the following results according to the cases of μ, k_1 and $\mathcal{N}$.
For $\mu \leq 1 - \Re(k_1) + \mathcal{N}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}|1-\mu-\Re(k_1)| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(1-\mu-\Re(k_1)) + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)} \\ &= \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(1-\mu-\Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)},\end{aligned}$$

and for $1 - \Re(k_1) + \mathcal{N} < \mu \leq 1 - \Re(k_1)$, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}|1-\mu-\Re(k_1)| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)}.\end{aligned}$$

Similarly, for $1 - \Re(k_1) < \mu < 1 - \Re(k_1) - \mathcal{N}$, we yield

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}|1-\mu-\Re(k_1)| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)},\end{aligned}$$

and finally, for $\mu \geq 1 - \Re(k_1) - \mathcal{N}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}|1-\mu-\Re(k_1)| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(\mu+\Re(k_1)-1) + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|) - 2B_1\right]}{4(1+2\lambda)} \\ &= \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(1-\mu-\Re(k_1)).\end{aligned}$$

Thus the proof is completed. □

Taking $\gamma = 1$, $\lambda = 0$ and $\varphi(z) = (1+Az)/(1+Bz)$ $(-1 \leq B < A \leq 1)$ in Theorems 2.3, 2.4 and 2.5, we have the following corollary.

Corollary 2.6. *If $f \in \mathcal{A}$ is given by (1.1) belongs to the class $\mathcal{S}_\sigma[A,B]$, then*

(1) *For $\mu \in \mathbb{C}$,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{A-B}{2} & \text{if } |B| + |4(1-\mu)(A-B) - B| < 2 \\ \frac{(A-B)}{4}\left[|B| + |4(1-\mu)(A-B) - B|\right] & \text{if } |B| + |4(1-\mu)(A-B) - B| \geq 2 \end{cases}.$$

(2) *For* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|B|(A-B)}{2} - (\mu - 1)\ \ (A-B)^2 & \textit{if} \quad \mu \leq 1 - \frac{1-|B|}{2(A-B)} \\ \frac{A-B}{2} & \textit{if} \quad 1 - \frac{1-|B|}{2(A-B)} < \mu < 1 + \frac{1-|B|}{2(A-B)} \\ \frac{|B|(A-B)}{2} + (\mu - 1)\ \ (A-B)^2 & \textit{if} \quad \mu \geq 1 + \frac{1-|B|}{2(A-B)} \end{cases}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} (A-B)\ \ \left[(A-B)(1-\mu) + \frac{|B|+B}{4}\right] & \textit{if} \ \ \mu \leq 1 + \frac{|B|+B-2}{4(A-B)} \\ \frac{A-B}{2} & \textit{if} \ \ 1 + \frac{|B|+B-2}{4(A-B)} < \mu < 1 - \frac{|B|-B-2}{4(A-B)} \\ (A-B)\ \ \left[(A-B)(\mu-1) + \frac{|B|-B}{4}\right] & \textit{if} \ \ \mu \geq 1 - \frac{|B|-B-2}{4(A-B)} \end{cases}.$$

Taking $\gamma = 1$, $\lambda = 1$ and $\varphi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorems 2.3, 2.4 and 2.5, we have the following corollary.

Corollary 2.7. *If* $f \in \mathcal{A}$ *is given by (1.1) belongs to the class* $\mathcal{C}_\sigma [A, B]$, *then*

(1) *For* $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{A-B}{6} & \textit{if} \ |B| + |3(1-\mu)(A-B) - B| < 2 \\ \frac{(A-B)}{12} \left[|B| + |3(1-\mu)(A-B) - B|\right] & \textit{if} \ |B| + |3(1-\mu)(A-B) - B| \geq 2 \end{cases}.$$

(2) *For* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|B|(A-B)}{6} - (\mu - 1)\ \frac{(A-B)^2}{4} & \textit{if} \quad \mu \leq 1 - \frac{2(1-|B|)}{3(A-B)} \\ \frac{A-B}{6} & \textit{if} \quad 1 - \frac{2(1-|B|)}{3(A-B)} < \mu < 1 + \frac{2(1-|B|)}{3(A-B)} \\ \frac{|B|(A-B)}{6} + (\mu - 1)\ \frac{(A-B)^2}{4} & \textit{if} \quad \mu \geq 1 + \frac{2(1-|B|)}{3(A-B)} \end{cases}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{A-B}{12}\ \ [3(A-B)(1-\mu) + |B| + B] & \textit{if} \ \ \mu \leq 1 + \frac{2|B|+2B-1}{6(A-B)} \\ \frac{A-B}{6} & \textit{if} \ \ 1 + \frac{2|B|+2B-1}{6(A-B)} < \mu < 1 - \frac{2|B|-2B-1}{6(A-B)} \\ \frac{A-B}{12}\ \ [3(A-B)(\mu-1) + |B| - B] & \textit{if} \ \ \mu \geq 1 - \frac{2|B|-2B-1}{6(A-B)} \end{cases}.$$

Taking $\gamma \in \mathbb{C}\backslash\{0\}$, $\lambda = 0$ and $\varphi(z) = (1+z)/(1-z)$ in Theorems 2.3, 2.4 and 2.5, then we have the following corollary.

Corollary 2.8. *If* $f \in \mathcal{A}$ *is given by (1.1) belongs to the class* $\mathcal{S}_\sigma^*[\gamma]$, *then*

(i) *For* $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} |\gamma| & \textit{if} \quad |1 + (1-\mu)\,8\gamma| < 1 \\ \frac{|\gamma|}{2}\ \ [|1 + (1-\mu)\,8\gamma| + 1] & \textit{if} \quad |1 + (1-\mu)\,8\gamma| \geq 1 \end{cases}.$$

(ii) *For* $\gamma > 0$ *and* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \gamma - 4(\mu - 1)\,\gamma^2 & \textit{if} \quad \mu \leq 1 \\ \gamma + 4(\mu - 1)\,\gamma^2 & \textit{if} \quad \mu > 1 \end{cases}.$$

(iii) *For* $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 4|\gamma|^2(1-\mu) + \frac{|\gamma|(1+|\sin\theta| - \cos\theta)}{2} & \textit{if} \ \ \mu \leq 1 + \chi_1(\gamma,\theta) \\ |\gamma| & \textit{if} \ \ 1 + \chi_1(\gamma,\theta) < \mu < 1 - \chi_2(\gamma,\theta) \\ \frac{|\gamma|(1+|\sin\theta| - \cos\theta)}{2} - 4|\gamma|^2(1-\mu) & \textit{if} \ \ \mu \geq 1 - \chi_2(\gamma,\theta) \end{cases}.$$

where $\chi_1(\gamma,\theta) = \frac{(|\sin\theta| - \cos\theta - 1)}{8|\gamma|}$ *and* $\chi_2(\gamma,\theta) = \frac{(|\sin\theta| + \cos\theta - 1)}{8|\gamma|}$.

Taking $\gamma \in \mathbb{C}\backslash\{0\}$, $\lambda = 1$ and $\varphi(z) = (1+z)/(1-z)$ in Theorems 2.3, 2.4 and 2.5, we obtain the following corollary.

Corollary 2.9. *If $f \in \mathcal{A}$ is given by (1.1) belongs to the class $\mathcal{C}_\sigma[\gamma]$, then*

(i) *For* $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{|\gamma|}{3} & \text{if } \left|1 + (1-\mu)\,6\gamma\right| < 1 \\ \frac{|\gamma|}{2}\left[\left|1 + (1-\mu)\,6\gamma\right| + 1\right] & \text{if } \left|1 + (1-\mu)\,6\gamma\right| \ge 1 \end{cases}.$$

(ii) *For* $\gamma > 0$ *and* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{\gamma}{3} - (\mu - 1)\,\gamma^2 & \text{if } \mu \le 1 \\ \frac{\gamma}{3} + (\mu - 1)\,\gamma^2 & \text{if } \mu > 1 \end{cases}.$$

(iii) *For* $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} |\gamma|^2(1-\mu) + \frac{|\gamma|(1+|\sin\theta| - \cos\theta)}{6} & \text{if } \mu \le 1 + \varphi_1(\gamma,\theta) \\ \frac{|\gamma|}{3} & \text{if } 1 + \varphi_1(\gamma,\theta) < \mu < 1 - \varphi_2(\gamma,\theta) \\ \frac{|\gamma|(1+|\sin\theta| - \cos\theta)}{6} - |\gamma|^2(1-\mu) & \text{if } \mu \ge 1 - \varphi_2(\gamma,\theta) \end{cases}$$

where $\varphi_1(\gamma,\theta) = \frac{(|\sin\theta| - \cos\theta - 1)}{6|\gamma|}$ *and* $\varphi_2(\gamma,\theta) = \frac{(|\sin\theta| + \cos\theta - 1)}{6|\gamma|}$.

Acknowledgment. The research of E. Deniz was supported by the Commission for the Scientific Research Projects of Kafkas University, project number 2016-FM-67.

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