



Second Hankel Determinant for a Certain Subclass of Bi-univalent Functions

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Abstract. In this paper, we introduce a subclass of analytic and bi-univalent functions in the open unit disk. Here, we give upper bound estimates for the second Hankel determinant of the functions that belong to this class. Some interesting applications and conclusions of the results obtained in this paper are also discussed.

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1. Introduction and Preliminaries

Let A denote the class of all complex valued functions $f(z)$ given by

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots = z + \sum_{n=2}^{\infty} a_nz^n, \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let S be the class of all functions in A which are univalent in U .

Some of the important and well-investigated subclasses of S include the classes $S^*(\alpha)$ and $K(\alpha)$ given below (see also [3, 5, 23]), such that $S^*(\alpha)$ is the class of starlike functions of order α ($0 \leq \alpha < 1$) and $K(\alpha)$ is the class of convex functions of order α ($0 \leq \alpha < 1$):

$$S^*(\alpha) = \left\{ f \in S : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in U \right\}, \quad \alpha \in [0, 1),$$

and

$$K(\alpha) = \left\{ f \in S : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in U \right\}, \quad \alpha \in [0, 1).$$

It is well known that (see [3]) every function $f \in S$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z, \quad z \in U$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq 1/4,$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \quad (1.2)$$

A function $f \in A$ is called bi-univalent in U if both f and f^{-1} are univalent in U . Let Σ denote the class of bi-univalent functions in U given by (1.1). Earlier, Brannan and Taha [1] introduced certain subclasses of bi-univalent function class Σ , namely bi-starlike function of order α denoted $S_{\Sigma}^*(\alpha)$ and bi-convex function of order α denoted $K_{\Sigma}(\alpha)$ corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, respectively. For each of the function classes, $S_{\Sigma}^*(\alpha)$ and $K_{\Sigma}(\alpha)$, non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ were found in [1, 24]. Many mathematicians (see [11, 13, 22, 25, 26]) have introduced and investigated several interesting subclasses of bi-univalent function class Σ and they have found upper bound estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Some mathematicians worked on coefficients of inverse functions of univalent functions. The works of Libera and Złotkiewicz [14, 15] are such studies. In [14], Libera and Złotkiewicz showed that some coefficients of the inverse of convex functions in the open unit disk of complex plane are bounded by 1. Their results are sharp. In [15], the same authors gave sharp upper estimates for the first six coefficients of the inverse of the univalent functions, such that the real part of their derivatives is positive. However, the coefficient problem for each of the Taylor–Maclaurin coefficients $|a_n|$, $n = 3, 4, \dots$ is still an open problem (see, for example [17, 19]).

An analytic function f is subordinate to an analytic function ϕ , written $f(z) \prec \phi(z)$, provided that there is an analytic function w defined on U with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = \phi(w(z))$. Ma and Minda [16] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general function. For this purpose, they considered an analytic function ϕ with positive real part in U , $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma–Minda starlike and Ma–Minda convex functions consists of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$, and $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$, respectively.

An analytic function f is bi-starlike of Ma–Minda type or bi-convex of Ma–Minda type if both f and f^{-1} are, respectively, Ma–Minda starlike or convex. These classes are denoted, respectively, by $S_{\Sigma}^*(\phi)$ and $K_{\Sigma}(\phi)$. In the sequel, it is assumed that ϕ is an analytic function with positive real part in U , satisfying $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the following form:

$$\phi(z) = 1 + b_1z + b_2z^2 + b_3z^3 + \dots, \quad b_1 > 0. \quad (1.3)$$

In 1976, Noonan and Thomas [18] defined the q th Hankel determinant of f for $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & \cdots & a_{n+q-1} \\ \cdot & \cdots & \cdot \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For $q = 2$ and $n = 1$, Fekete and Szegő [4] considered the Hankel determinant of f as $H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$. They made an earlier study for the estimates of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ with μ real. The well-known result due to them states that if $f \in A$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3 & \text{if } \mu \geq 1, \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right) & \text{if } 0 \leq \mu \leq 1, \\ 3 - 4\mu & \text{if } \mu \leq 0. \end{cases}$$

Furthermore, Hummel [7, 8] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is a convex function, and also Keogh and Merkes [12] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is a close-to-convex, starlike, and convex function in U . Moreover, we need to note that there are some studies by Kanas [9] and Srivastava et al. [21] on the Fekete–Szegő problem. In these studies, they obtained important results for the Fekete–Szegő problem. In [21], the authors solved the Fekete–Szegő problem for a subclass of close-to-convex functions. In [9], Kanas gave estimates for a unified Fekete–Szegő functional. In special cases, from the results of the study [9] by Kanas, a solution of the Fekete–Szegő problem can be obtained for some subclasses of analytic functions.

Recently, the upper bounds of $|H_2(2)| = |a_2 a_4 - a_3^2|$ for the classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ were obtained by Deniz et al. [2]. Very soon, Orhan et al. [20] reviewed the study of bounds for the second Hankel determinant of the subclass $M_\Sigma^\alpha(\beta)$ of bi-univalent functions.

Inspired by [2, 20, 27], we introduce a subclass of bi-univalent functions as follows.

Definition 1.1. A function $f \in \Sigma$ given by (1.1) is said to be in the class $M_\Sigma(\phi, \beta)$ ($\beta \geq 0$) if the following conditions are satisfied:

$$\begin{aligned} \left(\frac{zf'(z)}{f(z)}\right)^\beta \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\beta} &< \phi(z) \text{ and} \\ \left(\frac{zg'(w)}{g(w)}\right)^\beta \left(1 + \frac{zg''(w)}{g'(w)}\right)^{1-\beta} &< \phi(w), \end{aligned}$$

where ϕ is an analytic function given by (1.3) and $g = f^{-1}$.

Note that in a very early study by Kanas et al. [10], the subclass $M_\Sigma\left(\frac{1+z}{1-z}, \beta\right)$ [as a special case of the class $M_\Sigma(\phi, \beta)$] with $\phi(z) = \frac{1+z}{1-z}$ was introduced and studied.

Remark 1.2. By taking $\beta = 1$, we get $M_\Sigma(\phi, 1) = S_\Sigma^*(\phi)$.

Remark 1.3. By taking $\beta = 0$, we get $M_\Sigma(\phi, 0) = K_\Sigma(\phi)$.

Remark 1.4. Earlier, these classes $S_{\Sigma}^*(\phi)$ and $K_{\Sigma}(\phi)$ were introduced and studied by Ma and Minda [16].

The object of this paper is to determine the second Hankel determinant for the function class $M_{\Sigma}(\phi, \beta)$ and its special classes, and is to give upper bound estimate for $|H_2(2)|$.

To prove our main results, we shall need the following lemma.

Lemma 1.5. [3] *Let P be the class of all analytic functions $p(z)$ of the form*

$$p(z) = 1 + p_1z + p_2z^2 + \dots = 1 + \sum_{n=1}^{\infty} p_nz^n, \tag{1.4}$$

satisfying $Re(p(z)) > 0, z \in U$ and $p(0) = 1$. Then, $|p_n| \leq 2, n = 1, 2, 3, \dots$. This inequality is sharp for each n .

Moreover,

$$\begin{aligned} 2p_2 &= p_1^2 + (4 - p_1^2)x, \\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z, \end{aligned}$$

for some x, z with $|x| \leq 1, |z| \leq 1$.

2. Main Results

In this section, we prove the following theorem on upper bound estimate of the second Hankel determinant of the function class $M_{\Sigma}(\phi, \beta)$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $M_{\Sigma}(\phi, \beta)$ ($0 \leq \beta \leq 1$). Then,*

$$|a_2a_4 - a_3^2| \leq \begin{cases} H(2-) & \text{if } \Delta(\beta, b_1, b_2, b_3) \geq 0, \\ \max \left\{ \frac{b_1^2}{4(3-2\beta)^2}, H(2-) \right\} & \text{if } \begin{aligned} &c(\beta, b_1, b_2, b_3) \geq 0, \\ &\Delta(\beta, b_1, b_2, b_3) > 0, \end{aligned} \\ \frac{b_1^2}{4(3-2\beta)^2} & \text{if } \begin{aligned} &c(\beta, b_1, b_2, b_3) < 0, \\ &\Delta(\beta, b_1, b_2, b_3) \leq 0, \end{aligned} \\ \max \{H(t_0), H(2-)\} & \text{if } \begin{aligned} &c(\beta, b_1, b_2, b_3) \leq 0, \\ &\Delta(\beta, b_1, b_2, b_3) < 0, \end{aligned} \\ &c(\beta, b_1, b_2, b_3) > 0, \end{cases}$$

where

$$H(2-) = \frac{|\beta^3 - 3\beta^2 + 8\beta - 12|b_1^4 + 6b_1|2b_1 - 4b_2 + b_3|(2 - \beta)^3}{18(4 - 3\beta)(2 - \beta)^4},$$

$$H(t_0) = \frac{b_1^2}{4(3 - 2\beta)^2} - \frac{c^2(\beta, b_1, b_2, b_3)}{576(4 - 3\beta)(3 - 2\beta)^2(2 - \beta)^4\Delta(\beta, b_1, b_2, b_3)},$$

$$t_0 = \sqrt{\frac{-c(\beta, b_1, b_2, b_3)}{\Delta(\beta, b_1, b_2, b_3)}},$$

$$\Delta(\beta, b_1, b_2, b_3) = 2(3 - 2\beta)^2|\beta^3 - 3\beta^2 + 8\beta - 12|b_1^4 + 12(3 - 2\beta)^2(2 - \beta)^3b_1 \times (|2b_1 - 4b_2 + b_3| - b_1 - 2|b_2|) + 9(4 - 3\beta)(2 - \beta)^2b_1^2((2 - \beta)^2 - (3 - 2\beta)b_1),$$

$$c(\beta, b_1, b_2, b_3) = 6b_1(2 - \beta)^2(3(3 - 2\beta)(4 - 3\beta)b_1^2 + 8|b_2|(2 - \beta)(3 - 2\beta)^2 + 4b_1(2 - \beta)(3 - 2\beta)^2 - 6b_1(4 - 3\beta)(2 - \beta)^2).$$

Proof. Let $f \in M_\Sigma(\phi, \beta)$ and $g = f^{-1}$. Then, according to Definition 1.1, there are analytic functions $u, v : U \rightarrow U$, with $u(0) = 0 = v(0)$, satisfying

$$\begin{aligned} \left(\frac{zf'(z)}{f(z)}\right)^\beta \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\beta} &= \phi(u(z)) \text{ and} \\ \left(\frac{wg'(w)}{g(w)}\right)^\beta \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} &= \phi(v(w)). \end{aligned} \tag{2.1}$$

Let also the functions $p, q \in P$ be defined as follows:

$$p(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + \sum_{n=1}^\infty p_n z^n \text{ and } q(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + \sum_{n=1}^\infty q_n z^n.$$

It follows that:

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left(p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) z^3 + \dots \right) \tag{2.2}$$

and

$$v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left(q_1 z + \left(q_2 - \frac{q_1^2}{2} \right) z^2 + \left(q_3 - q_1 q_2 + \frac{q_1^3}{4} \right) z^3 + \dots \right). \tag{2.3}$$

From (2.2) and (2.3), considering (1.3), we can easily show that

$$\begin{aligned} \phi(u(z)) &= 1 + \frac{b_1 p_1}{2} z + \left(\frac{b_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} b_2 p_1^2 \right) z^2 \\ &+ \left(\frac{b_1}{2} \left(p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) + \frac{b_2 p_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{b_3 p_1^3}{8} \right) z^3 + \dots \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \phi(v(w)) &= 1 + \frac{b_1 q_1}{2} w + \left(\frac{b_1}{2} \left(q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} b_2 q_1^2 \right) w^2 \\ &+ \left(\frac{b_1}{2} \left(q_3 - q_1 q_2 + \frac{q_1^3}{4} \right) + \frac{b_2 q_1}{2} \left(q_2 - \frac{q_1^2}{2} \right) + \frac{b_3 q_1^3}{8} \right) w^3 + \dots \end{aligned} \tag{2.5}$$

By substituting the expressions (2.4) and (2.5) in (2.1), and equating the coefficients, we get

$$(2 - \beta) a_2 = \frac{b_1 p_1}{2}, \tag{2.6}$$

$$2(3 - 2\beta) a_3 + \frac{1}{2} ((\beta - 2)^2 - 3(4 - 3\beta)) a_2^2 = \frac{b_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} b_2 p_1^2, \tag{2.7}$$

$$\begin{aligned} 3(4 - 3\beta) a_4 + ((2\beta - 3)^2 + 23\beta - 27) a_2 a_3 - \frac{1}{6} (\beta^3 + 21\beta^2 + 20\beta - 48) a_2^3 \\ = \frac{b_1}{2} \left(p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) + \frac{b_2 p_1}{2} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{b_3 p_1^3}{8} \end{aligned} \tag{2.8}$$

and

$$-(2 - \beta)a_2 = \frac{b_1q_1}{2}, \tag{2.9}$$

$$-2(3 - 2\beta)a_3 + \frac{1}{2}(\beta^2 - 11\beta + 16)a_2^2 = \frac{b_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}b_2q_1^2, \tag{2.10}$$

$$\begin{aligned} -3(4 - 3\beta)a_4 + (4\beta^2 - 34\beta + 42)a_2a_3 + \frac{1}{6}(\beta^3 - 27\beta^2 + 158\beta - 192)a_2^3 \\ = \frac{b_1}{2}\left(q_3 - q_1q_2 + \frac{q_1^3}{4}\right) + \frac{b_2q_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{b_3q_1^3}{8}. \end{aligned} \tag{2.11}$$

From (2.6) and (2.9), we obtain

$$a_2 = \frac{b_1p_1}{2(2 - \beta)} = \frac{-b_1q_1}{2(2 - \beta)}, \tag{2.12}$$

which is equivalent to

$$p_1 = -q_1. \tag{2.13}$$

From (2.7) and (2.10), considering (2.12), we can easily obtain that

$$a_3 = \frac{b_1^2p_1^2}{4(2 - \beta)^2} + \frac{b_1(p_2 - q_2)}{8(3 - 2\beta)}. \tag{2.14}$$

On the other hand, subtracting (2.11) from (2.8) and considering (2.12) and (2.14), we get

$$\begin{aligned} a_4 = \frac{b_1^3p_1^3(\beta^3 - 3\beta^2 - 46\beta + 60)}{144(2 - \beta)^3(4 - 3\beta)} + \frac{5b_1^2p_1(p_2 - q_2)}{32(2 - \beta)(3 - 2\beta)} + \frac{b_1(p_3 - q_3)}{12(4 - 3\beta)} \\ + \frac{(b_2 - b_1)p_1(p_2 + q_2)}{12(4 - 3\beta)} - \frac{p_1^3(b_1 - 2b_2 + b_3)}{24(4 - 3\beta)}. \end{aligned} \tag{2.15}$$

Thus, from (2.12), (2.14), and (2.15), we established that

$$\begin{aligned} a_2a_4 - a_3^2 = \frac{(\beta^3 - 3\beta^2 + 8\beta - 12)b_1^4 - 6(2 - \beta)^3b_1(b_1 - 2b_2 + b_3)}{288(2 - \beta)^4(4 - 3\beta)}p_1^4 - \frac{b_1^2(p_2 - q_2)^2}{64(3 - 2\beta)^2} \\ + \frac{b_1^3p_1^2(p_2 - q_2)}{64(3 - 2\beta)(2 - \beta)^2} + \frac{b_1p_1^2(b_2 - b_1)(p_2 + q_2)}{24(4 - 3\beta)(2 - \beta)} + \frac{b_1^2p_1(p_3 - q_3)}{24(4 - 3\beta)(2 - \beta)}. \end{aligned} \tag{2.16}$$

According to Lemma 1.1, we have

$$2p_2 = p_1^2 + (4 - p_1^2)x \quad \text{and} \quad 2q_2 = q_1^2 + (4 - q_1^2)y \tag{2.17}$$

and

$$\begin{aligned} 4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z, \\ 4q_3 = q_1^3 + 2(4 - q_1^2)q_1y - (4 - q_1^2)q_1y^2 + 2(4 - q_1^2)(1 - |y|^2)w, \end{aligned} \tag{2.18}$$

for some x, y, z , and w with $|x| \leq 1, |y| \leq 1, |z| \leq 1$, and $|w| \leq 1$.

Since $p_1 = -q_1$, from (2.17) and (2.18), we get

$$p_2 - q_2 = \frac{4 - p_1^2}{2}(x - y), \quad p_2 + q_2 = p_1^2 + \frac{4 - p_1^2}{2}(x + y) \tag{2.19}$$

and

$$\begin{aligned} p_3 - q_3 = \frac{p_1^3}{2} + \frac{p_1(4 - p_1^2)}{2}(x + y) - \frac{p_1(4 - p_1^2)}{4}(x^2 + y^2) \\ + \frac{4 - p_1^2}{2}\left((1 - |x|^2)z - (1 - |y|^2)w\right). \end{aligned} \tag{2.20}$$

According to Lemma 1.1, we may assume without any restriction that $t \in [0, 2]$, where $t = |p_1|$.

Thus, substituting the expressions (2.19) and (2.20) in (2.16) and using the triangle inequality, letting $|x| = \zeta, |y| = \eta$, we obtain

$$|a_2a_4 - a_3^2| \leq c_1 + c_2(\zeta + \eta) + c_3(\zeta^2 + \eta^2) + c_4(\zeta + \eta)^2 := F(\zeta, \eta), \tag{2.21}$$

where

$$c_1 = c_1(t) := \frac{t(|\beta^3 - 3\beta^2 + 8\beta - 12|b_1^4t^3 + 6(2-\beta)^3b_1|2b_1 - 4b_2 + b_3|t^3 + 12b_1^2(4-t^2)(2-\beta)^3)}{288(2-\beta)^4(4-3\beta)} \geq 0,$$

$$c_2 = c_2(t) := \frac{b_1(4-t^2)t^2(3b_1^2(4-3\beta) + 8|b_2|(2-\beta)(3-2\beta))}{384(2-\beta)^2(3-2\beta)(4-3\beta)} \geq 0,$$

$$c_3 = c_3(t) := \frac{b_1^2(4-t^2)(t-2)t}{96(2-\beta)(4-3\beta)} \leq 0, \quad c_4 = c_4(t) := \frac{b_1^2(4-t^2)^2}{256(3-2\beta)} \geq 0.$$

Now, we need to maximize the function $F(\zeta, \eta)$ in the closed square $\Omega = \{(\zeta, \eta) : \zeta, \eta \in [0, 1]\}$ for $t \in [0, 2]$. Since the coefficients of the function $F(\zeta, \eta)$ are dependent to variable t , we must investigate the maximum of $F(\zeta, \eta)$ according to t taking into account these cases $t = 0, t = 2$, and $t \in (0, 2)$.

Let $t = 0$. Then, we write

$$F(\zeta, \eta) = \frac{b_1^2}{16(3-2\beta)^2}(\zeta + \eta)^2.$$

We can easily see that the maximum of the function $F(\zeta, \eta)$ occurs at $(\zeta, \eta) = (1, 1)$ and

$$\max \{F(\zeta, \eta) : \zeta, \eta \in [0, 1]\} = F(1, 1) = \frac{b_1^2}{4(3-2\beta)^2}. \tag{2.22}$$

Now, let $t = 2$. In this case, $F(\zeta, \eta)$ is a constant function as follows:

$$F(\zeta, \eta) = \frac{|\beta^3 - 3\beta^2 + 8\beta - 12|b_1^4 + 6(2-\beta)^3b_1|2b_1 - 4b_2 + b_3|}{18(2-\beta)^4(4-3\beta)}. \tag{2.23}$$

In the case $t \in (0, 2)$, we will examine the maximum of the function $F(\zeta, \eta)$ taking into account the sign of $F_{\zeta\zeta}(\zeta, \eta)F_{\eta\eta}(\zeta, \eta) - [F_{\zeta\eta}(\zeta, \eta)]^2$.

By simple computation, we can easily see that

$$F_{\zeta\zeta}(\zeta, \eta)F_{\eta\eta}(\zeta, \eta) - [F_{\zeta\eta}(\zeta, \eta)]^2 = 4c_3(t)[c_3(t) + 2c_4(t)].$$

Since $c_3(t) < 0$ and

$$c_3(t) + 2c_4(t) = \frac{b_1^2(4-t^2)(2-t)}{384(2-\beta)(3-2\beta)(4-3\beta)}\Psi(t),$$

where $\Psi(t) = A(\beta)t + B(\beta) > 0$ for all $t \in (0, 2)$ and $\beta \in [0, 1]$, and

$$A(\beta) = 3(2-\beta)(4-3\beta) - 4(3-2\beta) = 9\beta^2 - 22\beta + 12,$$

$$B(\beta) = 6(2-\beta)(4-3\beta) = 18\beta^2 - 60\beta + 48,$$

we conclude that

$$F_{\zeta\zeta}(\zeta, \eta)F_{\eta\eta}(\zeta, \eta) - [F_{\zeta\eta}(\zeta, \eta)]^2 < 0$$

for all $(\zeta, \eta) \in \Omega$.

Thus, the function $F(\zeta, \eta)$ cannot have a critical point in Ω . Consequently, the function $F(\zeta, \eta)$ cannot have a local maximum in Ω . Therefore, we must investigate the maximum of the function $F(\zeta, \eta)$ on the boundary of the square Ω .

For $\zeta = 0$ and $0 \leq \eta \leq 1$ (the case $\eta = 0$ and $0 \leq \zeta \leq 1$ is examined in a similar manner), we write

$$F(0, \eta) = c_1(t) + c_2(t)\eta + [c_3(t) + c_4(t)]\eta^2 := \Phi_1(\eta).$$

By simple computation, we have

$$\Phi_1'(\eta) = c_2(t) + 2[c_3(t) + c_4(t)]\eta.$$

Now, we will examine the sign of the function $\Phi_1'(\eta)$ depending on the different cases of the sign of $c_3(t) + c_4(t)$ as follows.

(i) Let $c_3(t) + c_4(t) \geq 0$, then $\Phi_1'(\eta) > 0$. That is, $\Phi_1(\eta)$ is an increasing function.

(ii) Let $c_3(t) + c_4(t) < 0$. In this case, it is clear that $c_2(t) + 2[c_3(t) + c_4(t)]\eta \geq c_2(t) + 2[c_3(t) + c_4(t)]$ for all $t \in (0, 2)$ and $\eta \in [0, 1]$. In addition, we can easily show that $c_2(t) + 2[c_3(t) + c_4(t)] > 0$ for all $t \in (0, 2)$. Thus, we conclude that $\Phi_1'(\eta) > 0$. That is, $\Phi_1(\eta)$ is an increasing function.

Consequently, in both cases:

$$\begin{aligned} \max \{F(0, \eta) : \eta \in [0, 1]\} &= \max \{\Phi_1(\eta) : \eta \in [0, 1]\} \\ &= \Phi_1(1) = c_1(t) + c_2(t) + c_3(t) + c_4(t). \end{aligned}$$

For $\zeta = 1$ and $0 \leq \eta \leq 1$ (the case $\eta = 1$ and $0 \leq \zeta \leq 1$ is examined in a similar manner), we write

$$\begin{aligned} F(1, \eta) &= c_1(t) + c_2(t) + c_3(t) + c_4(t) + [c_2(t) + 2c_4(t)]\eta + [c_3(t) + c_4(t)]\eta^2 \\ &:= \Phi_2(\eta). \end{aligned}$$

Similarly to the previous case, we can easily show that $\Phi_2(\eta)$ is an increasing function.

Therefore,

$$\begin{aligned} \max \{F(1, \eta) : \eta \in [0, 1]\} &= \max \{\Phi_2(\eta) : \eta \in [0, 1]\} \\ &= \Phi_2(1) = c_1(t) + 2[c_2(t) + c_3(t)] + 4c_4(t). \end{aligned}$$

On the other hand, the following inequality holds for all $t \in (0, 2)$:

$$c_1(t) + 2[c_2(t) + c_3(t)] + 4c_4(t) > c_1(t) + c_2(t) + c_3(t) + c_4(t).$$

Thus,

$$\max \{F(\zeta, \eta) : \zeta, \eta \in [0, 1]\} = F(1, 1) = c_1(t) + 2[c_2(t) + c_3(t)] + 4c_4(t). \tag{2.24}$$

Since $\Phi_1(1) < \Phi_2(1)$ for all $t \in [0, 2]$,

$$\max \{F(\zeta, \eta) : \zeta, \eta \in [0, 1]\} = F(1, 1) = \Phi_2(1).$$

Thus, the maximum of the function $F(\zeta, \eta)$ occurs at $(\zeta, \eta) = (1, 1)$.

Let us define the function $H : (0, 2) \rightarrow \mathbb{R}$ as follows:

$$H(t) := c_1(t) + 2(c_2(t) + c_3(t)) + 4c_4(t). \tag{2.25}$$

Substituting the value $c_j(t)$, $j = 1, 2, 3, 4$ in the (2.25), we obtain

$$H(t) = \frac{b_1^2}{4(3 - 2\beta)^2} + \frac{\Delta(\beta, b_1, b_2, b_3)t^4 + 2c(\beta, b_1, b_2, b_3)t^2}{576(4 - 3\beta)(3 - 2\beta)^2(2 - \beta)^4},$$

where

$$\begin{aligned} \Delta(\beta, b_1, b_2, b_3) &= 2(3 - 2\beta)^2 |\beta^3 - 3\beta^2 + 8\beta - 12| b_1^4 + 12(3 - 2\beta)^2(2 - \beta)^3 b_1 \\ &\times (|2b_1 - 4b_2 + b_3| - b_1 - 2|b_2|) + 9(4 - 3\beta)(2 - \beta)^2 b_1^2 ((2 - \beta)^2 - (3 - 2\beta)b_1), \\ c(\beta, b_1, b_2, b_3) &= 6b_1(2 - \beta)^2 (3(3 - 2\beta)(4 - 3\beta)b_1^2 + 8|b_2|(2 - \beta)(3 - 2\beta)^2 \\ &+ 4b_1(2 - \beta)(3 - 2\beta)^2 - 6b_1(4 - 3\beta)(2 - \beta)^2). \end{aligned}$$

Now, we must investigate the maximum of the function $H(t)$ in the interval $(0, 2)$.

By simple computation, we can easily show that

$$H'(t) = \frac{\Delta(\beta, b_1, b_2, b_3)t^2 + c(\beta, b_1, b_2, b_3)}{144(4 - 3\beta)(3 - 2\beta)^2(2 - \beta)^4}t.$$

We will examine the sign of the function $H'(t)$ depending on the different cases of the signs of $\Delta(\beta, b_1, b_2, b_3)$ and $c(\beta, b_1, b_2, b_3)$ as follows.

(I) Let $\Delta(\beta, b_1, b_2, b_3) \geq 0$ and $c(\beta, b_1, b_2, b_3) \geq 0$, then $H'(t) \geq 0$, so $H(t)$ is an increasing function. Therefore,

$$\begin{aligned} &\max \{H(t) : t \in (0, 2)\} \\ &= H(2-) \\ &= \frac{|\beta^3 - 3\beta^2 + 8\beta - 12| b_1^4 + 6b_1 |2b_1 - 4b_2 + b_3| (2 - \beta)^3}{18(4 - 3\beta)(2 - \beta)^4}. \end{aligned} \tag{2.26}$$

That is,

$$\max \{ \max \{F(\zeta, \eta) : \zeta, \eta \in [0, 1]\} : t \in (0, 2) \} = H(2-).$$

(II) Let $\Delta(\beta, b_1, b_2, b_3) > 0$ and $c(\beta, b_1, b_2, b_3) < 0$, then

$t_0 = \sqrt{\frac{-c(\beta, b_1, b_2, b_3)}{\Delta(\beta, b_1, b_2, b_3)}}$ is a critical point of the function $H(t)$. We assume that $t_0 \in (0, 2)$. Since $H''(t_0) > 0$, t_0 is a local minimum point of the function $H(t)$. That is, the function $H(t)$ cannot have a local maximum.

(III) Let $\Delta(\beta, b_1, b_2, b_3) \leq 0$ and $c(\beta, b_1, b_2, b_3) \leq 0$, then $H'(t) \leq 0$. Thus, $H(t)$ is an decreasing function on the interval $(0, 2)$. Therefore,

$$\max \{H(t) : t \in (0, 2)\} = H(0+) = \frac{b_1^2}{4(3 - 2\beta)^2}. \tag{2.27}$$

(IV) Let $\Delta(\beta, b_1, b_2, b_3) < 0$ and $c(\beta, b_1, b_2, b_3) > 0$, then t_0 is a critical point of the function $H(t)$. We assume that $t_0 \in (0, 2)$. Since $H''(t_0) < 0$, t_0 is a local maximum point of the function $H(t)$ and the maximum value occurs at $t = t_0$.

Therefore,

$$\max \{H(t) : t \in (0, 2)\} = H(t_0), \tag{2.28}$$

where

$$H(t_0) = \frac{b_1^2}{4(3 - 2\beta)^2} - \frac{c^2(\beta, b_1, b_2, b_3)}{576(4 - 3\beta)(3 - 2\beta)^2(2 - \beta)^4\Delta(\beta, b_1, b_2, b_3)}.$$

Thus, from (2.22) to (2.28), the proof of Theorem 2.1 is completed. \square

In the special cases from Theorem 2.1, we arrive at the following results.

Corollary 2.2. *Let the function $f(z)$ given by (1.1) be in the class*

$$M_\Sigma(\phi, \beta) \ (0 \leq \beta \leq 1), \ b_1 = 2|b_2| < 1, \ \text{and} \\ |2b_1 - 4b_2 + b_3| - b_1 - 2|b_2| \geq 0. \ \text{Then,}$$

$$|a_2a_4 - a_3^2| \leq \frac{|\beta^3 - 3\beta^2 + 8\beta - 12| b_1^4 + 6b_1 |2b_1 - 4b_2 + b_3| (2 - \beta)^3}{18(4 - 3\beta)(2 - \beta)^4}.$$

In particular if $2b_1 = 4b_2 = 1$ and $b_3 = 1$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{6(4 - 3\beta)(2 - \beta)} + \frac{|\beta^3 - 3\beta^2 + 8\beta - 12|}{288(4 - 3\beta)(2 - \beta)^4}.$$

Corollary 2.3. *Let the function $f(z)$ given by (1.1) be in the class*

$$M_\Sigma(\phi, \beta) \ (0 \leq \beta \leq 1), \ b_1 < 1, \ b_1 \neq 2|b_2|, \ \text{and} \\ |2b_1 - 4b_2 + b_3| - b_1 - 2|b_2| \geq 0. \ \text{Then,}$$

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{b_1^2}{4(3 - 2\beta)^2}, \frac{|\beta^3 - 3\beta^2 + 8\beta - 12| b_1^4 + 6b_1 |2b_1 - 4b_2 + b_3| (2 - \beta)^3}{18(4 - 3\beta)(2 - \beta)^4} \right\}.$$

In particular if $2b_1 = 2b_2 = 1$ and $b_3 = 2$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{6(4 - 3\beta)(2 - \beta)} + \frac{|\beta^3 - 3\beta^2 + 8\beta - 12|}{288(4 - 3\beta)(2 - \beta)^4}.$$

Corollary 2.4. *Let the function $f(z)$ given by (1.1) be in the class*

$$M_\Sigma(\phi, \beta) \ (0 \leq \beta \leq 1), \ b_1 \geq 1 \ \text{and} \\ |2b_1 - 4b_2 + b_3| - b_1 - 2|b_2| \geq 0 \ (\text{or } |2b_1 - 4b_2 + b_3| - b_1 - 2|b_2| < 0).$$

Then,

$$|a_2a_4 - a_3^2| \leq \frac{b_1^2}{4(3 - 2\beta)^2}.$$

In particular if $b_1 = 1$ and $|2b_1 - 4b_2 + b_3| - b_1 - 2|b_2| \geq 1$ (or $|2b_1 - 4b_2 + b_3| - b_1 - 2|b_2| < 1$), then

$$|a_2a_4 - a_3^2| \leq \frac{1}{4(3 - 2\beta)^2}.$$

Corollary 2.5. *Let the function $f(z)$ given by (1.1) be in the class*

$$M_\Sigma(\phi, \beta) \ (0 \leq \beta \leq 1), \ b_1 = 2|b_2| \geq 1 \ \text{and}$$

$|2b_1 - 4b_2 + b_3| - b_1 - 2|b_2| \geq 0$ (or $|2b_1 - 4b_2 + b_3| - b_1 - 2|b_2| < 0$).
 Then,

$$|a_2a_4 - a_3^2| \leq \max \left\{ H(t_0), \frac{|\beta^3 - 3\beta^2 + 8\beta - 12| b_1^4 + 6b_1 |2b_1 - 4b_2 + b_3| (2 - \beta)^3}{18(4 - 3\beta)(2 - \beta)^4} \right\}.$$

In particular if $b_1 = 2b_2 = 1$ and $b_3 = 4$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{3(4 - 3\beta)(2 - \beta)} + \frac{|\beta^3 - 3\beta^2 + 8\beta - 12|}{18(4 - 3\beta)(2 - \beta)^4}.$$

The following theorems are the direct results of Theorem 2.1.

Theorem 2.6. Let the function $f(z)$ given by (1.1) be in the class $S_{\Sigma}^*(\phi)$.

1. If b_1, b_2 , and b_3 satisfy the conditions:

$$4b_1^3 - 3b_1^2 - b_1 + 4|2b_1 - 4b_2 + b_3| - 8|b_2| \geq 0, \quad 3b_1^2 - 2b_1 + 8|b_2| \geq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{b_1 (b_1^3 + |2b_1 - 4b_2 + b_3|)}{3}.$$

2. If b_1, b_2 , and b_3 satisfy the conditions

$$4b_1^3 - 3b_1^2 - b_1 + 4|2b_1 - 4b_2 + b_3| - 8|b_2| > 0, \quad 3b_1^2 - 2b_1 + 8|b_2| < 0,$$

then

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{b_1^2}{4}, \frac{b_1 (b_1^3 + |2b_1 - 4b_2 + b_3|)}{3} \right\}.$$

3. If b_1, b_2 , and b_3 satisfy the conditions

$$4b_1^3 - 3b_1^2 - b_1 + 4|2b_1 - 4b_2 + b_3| - 8|b_2| \leq 0, \quad 3b_1^2 - 2b_1 + 8|b_2| \leq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{b_1^2}{4}.$$

4. If b_1, b_2 , and b_3 satisfy the conditions

$$4b_1^3 - 3b_1^2 - b_1 + 4|2b_1 - 4b_2 + b_3| - 8|b_2| < 0, \quad 3b_1^2 - 2b_1 + 8|b_2| > 0,$$

then

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{b_1 (b_1^3 + |2b_1 - 4b_2 + b_3|)}{3}, \frac{b_1^2}{4} - \frac{b_1 (3b_1^2 - 2b_1 + 8|b_2|)^2}{48(4b_1^3 - b_1^2 - b_1 + 4|2b_1 - 4b_2 + b_3| - 8|b_2|)} \right\}.$$

Theorem 2.7. Let the function $f(z)$ given by (1.1) be in the class $K_{\Sigma}(\phi)$.

1. If b_1, b_2 , and b_3 satisfy the conditions:

$$3b_1^3 - 6b_1^2 - 4b_1 + 12|2b_1 - 4b_2 + b_3| - 24|b_2| \geq 0, \quad 3b_1^2 - 2b_1 + 12|b_2| \geq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{b_1 (b_1^3 + 4 |2b_1 - 4b_2 + b_3|)}{96}.$$

2. If $b_1, b_2,$ and b_3 satisfy the conditions:

$$3b_1^3 - 6b_1^2 - 4b_1 + 12 |2b_1 - 4b_2 + b_3| - 24 |b_2| > 0, \quad 3b_1^2 - 2b_1 + 12 |b_2| < 0,$$

then

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{b_1^2}{36}, \frac{b_1 (b_1^3 + 4 |2b_1 - 4b_2 + b_3|)}{96} \right\}.$$

3. If $b_1, b_2,$ and b_3 satisfy the conditions

$$3b_1^3 - 6b_1^2 - 4b_1 + 12 |2b_1 - 4b_2 + b_3| - 24 |b_2| \leq 0, \quad 3b_1^2 - 2b_1 + 12 |b_2| \leq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{b_1^2}{36}.$$

4. If $b_1, b_2,$ and b_3 satisfy the conditions

$$3b_1^3 - 6b_1^2 - 4b_1 + 12 |2b_1 - 4b_2 + b_3| - 24 |b_2| < 0, \quad 3b_1^2 - 2b_1 + 12 |b_2| > 0,$$

then

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{b_1 (b_1^3 + 4 |2b_1 - 4b_2 + b_3|)}{96}, \frac{b_1^2}{36} - \frac{b_1 (3b_1^2 - 2b_1 + 12 |b_2|)^2}{288 (3b_1^3 - 6b_1^2 - 4b_1 + 12 |2b_1 - 4b_2 + b_3| - 24 |b_2|)} \right\}.$$

3. Concluding Remarks

If the function $\phi(z)$ is given by

$$\phi(z) := \frac{1 + az}{1 + bz} = 1 + (a - b) \sum_{n=1}^{\infty} (-b)^{n-1} z^n \quad (-1 \leq b < a \leq 1), \quad (3.1)$$

then $b_1 = a - b, b_2 = -b(a - b),$ and $b_3 = b^2(a - b).$

By taking $a = 1 - 2\alpha$ and $b = -1$ in (3.1), we get

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha) \sum_{n=1}^{\infty} z^n \quad (0 \leq \alpha < 1). \quad (3.2)$$

Hence, $b_1 = b_2 = b_3 = 2(1 - \alpha).$

Furthermore, by taking $\alpha = 0$ in (3.2), we get

$$\phi(z) = \frac{1 + z}{1 - z} = 1 + 2 \sum_{n=1}^{\infty} z^n. \quad (3.3)$$

Hence, $b_1 = b_2 = b_3 = 2.$

By choosing $\phi(z)$ of the form (3.1) in Theorem 2.1, we arrive at the following corollary.

Corollary 3.1. *Let the function $f(z)$ given by (1.1) be in the class $M_\Sigma\left(\frac{1+az}{1+bz}, \beta\right)$ ($0 \leq \beta \leq 1$).*

1. *If a and b satisfy the conditions: $\Delta(\beta, a, b) \geq 0$, $c(\beta, a, b) \geq 0$, then*

$$|a_2a_4 - a_3^2| \leq \frac{|\beta^3 - 3\beta^2 + 8\beta - 12|(a - b)^4 + 6|b^2 + 4b + 2|(a - b)^2(2 - \beta)^3}{18(4 - 3\beta)(2 - \beta)^4}.$$

2. *If a and b satisfy the conditions: $\Delta(\beta, a, b) > 0$, $c(\beta, a, b) < 0$, then*

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{(a - b)^2}{4(3 - 2\alpha)^2}, \frac{|\beta^3 - 3\beta^2 + 8\beta - 12|(a - b)^4 + 6|b^2 + 4b + 2|(a - b)^2(2 - \beta)^3}{18(4 - 3\beta)(2 - \beta)^4} \right\}.$$

3. *If a and b satisfy the conditions: $\Delta(\beta, a, b) \leq 0$, $c(\beta, a, b) \leq 0$, then*

$$|a_2a_4 - a_3^2| \leq \frac{(a - b)^2}{4(3 - 2\beta)^2}.$$

4 *If a and b satisfy the conditions: $\Delta(\alpha, a, b) < 0$, $c(\alpha, a, b) > 0$, then*

$$|a_2a_4 - a_3^2| \leq \max \left\{ H(t_0), \frac{|\beta^3 - 3\beta^2 + 8\beta - 12|(a - b)^4 + 6|b^2 + 4b + 2|(a - b)^2(2 - \beta)^3}{18(4 - 3\beta)(2 - \beta)^4} \right\}.$$

Here,

$$H(t_0) = \frac{(a - b)^2}{4(3 - 2\beta)^2} - \frac{c^2(\beta, a, b)}{576(4 - 3\beta)(3 - 2\beta)^2(2 - \beta)^4\Delta(\beta, a, b)},$$

$$t_0 = \sqrt{\frac{-c(\beta, a, b)}{\Delta(\beta, a, b)}},$$

$$\Delta(\beta, a, b) = (a - b)^2 \left(\begin{aligned} &2(3 - 2\beta)^2|\beta^3 - 3\beta^2 + 8\beta - 12|(a - b)^2 \\ &+ 12(3 - 2\beta)^2(2 - \beta)^3(|b^2 + 4b + 2| - 2|b| - 1) \\ &+ 9(4 - 3\beta)(2 - \beta)^2((2 - \beta)^2 - (3 - 2\beta)(a - b)) \end{aligned} \right)$$

$$c(\beta, a, b) = 6(a - b)^2(2 - \beta)^2(3(3 - 2\beta)(4 - 3\beta)(a - b) + 8|b|(2 - \beta)(3 - 2\beta)^2 - 2(2 - \beta)(\beta^2 - 6\beta + 6)).$$

By taking $\phi(z)$ of the form (3.1) in Theorems 2.6 and 2.7, we can easily give the following corollaries.

Corollary 3.2. *Let the function $f(z)$ given by (1.1) be in the class $S_\Sigma^*\left(\frac{1+az}{1+bz}\right)$.*

1. *If a and b satisfy the conditions*

$$4(a - b)^2 - 3(a - b) + 4|b^2 + 4b + 2| - 8|b| - 1 \geq 0, \quad 3(a - b) + 8|b| - 2 \geq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{(a - b)^2}{3} ((a - b)^2 + |b^2 + 4b + 2|).$$

2. *If a and b satisfy the conditions*

$$4(a - b)^2 - 3(a - b) + 4|b^2 + 4b + 2| - 8|b| - 1 > 0, \quad 3(a - b) + 8|b| - 2 < 0,$$

then

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{(a-b)^2}{4}, \frac{(a-b)^2}{3} ((a-b)^2 + |b^2 + 4b + 2|) \right\}.$$

3. If a and b satisfy the conditions

$$4(a-b)^2 - 3(a-b) + 4|b^2 + 4b + 2| - 8|b| - 1 \leq 0, \quad 3(a-b) + 8|b| - 2 \leq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{(a-b)^2}{4}.$$

4. If a and b satisfy the conditions

$$4(a-b)^2 - 3(a-b) + 4|b^2 + 4b + 2| - 8|b| - 1 < 0, \quad 3(a-b) + 8|b| - 2 > 0,$$

then

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{(a-b)^2}{3} ((a-b)^2 + |b^2 + 4b + 2|), \frac{(a-b)^2}{4} - \frac{(a-b)^2(3(a-b)+8|b|-2)^2}{48(4(a-b)^2-3(a-b)+4|b^2+4b+2|-8|b|-1)} \right\}.$$

Corollary 3.3. Let the function $f(z)$ given by (1.1) be in the class $K_\Sigma \left(\frac{1+a_z}{1+b_z} \right)$.

1. If a and b satisfy the conditions

$$3(a-b)^2 - 6(a-b) + 12|b^2 + 4b + 2| - 24|b| - 4 \geq 0, \quad 3(a-b) + 12|b| - 2 \geq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{(a-b)^2}{96} ((a-b)^2 + 4|b^2 + 4b + 2|).$$

2. If a and b satisfy the conditions

$$3(a-b)^2 - 6(a-b) + 12|b^2 + 4b + 2| - 24|b| - 4 > 0, \quad 3(a-b) + 12|b| - 2 < 0,$$

then

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{(a-b)^2}{36}, \frac{(a-b)^2}{96} ((a-b)^2 + 4|b^2 + 4b + 2|) \right\}.$$

3. If a and b satisfy the conditions

$$3(a-b)^2 - 6(a-b) + 12|b^2 + 4b + 2| - 24|b| - 4 \leq 0, \quad 3(a-b) + 12|b| - 2 \leq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{(a-b)^2}{36}.$$

4. If a and b satisfy the conditions

$$3(a-b)^2 - 6(a-b) + 12|b^2 + 4b + 2| - 24|b| - 4 < 0, \quad 3(a-b) + 12|b| - 2 > 0,$$

then

$$|a_2a_4 - a_3^2| \leq \max \left\{ \frac{(a-b)^2}{96} \left((a-b)^2 + |b^2 + 4b + 2| \right), \frac{(a-b)^2}{6} - \frac{(a-b)^2(3(a-b)+8|b|-2)^2}{288(3(a-b)^2-6(a-b)+12|b^2+4b+2|-24|b|-4)} \right\}.$$

By choosing $\phi(z)$ of the form (3.2) in Theorem 2.1, we have the following corollary.

Corollary 3.4. *Let the function $f(z)$ given by (1.1) be in the class*

$$M_\Sigma \left(\frac{1+(1-2\alpha)z}{1-z}, \beta \right) = M_\Sigma (\alpha, \beta) \quad (0 \leq \alpha < 1, 0 \leq \beta \leq 1). \text{ Then,}$$

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{8|\beta^3-3\beta^2+8\beta-12|(1-\alpha)^4+12(2-\beta)^3(1-\alpha)^2}{9(2-\beta)^4(4-3\beta)}, & \alpha \in [0, 1-\alpha_0], \\ \max \left\{ H(t_0), \frac{8|\beta^3-3\beta^2+8\beta-12|(1-\alpha)^4+12(2-\beta)^3(1-\alpha)^2}{9(2-\beta)^4(4-3\beta)} \right\}, & \alpha \in (1-\alpha_0, 1), \end{cases}$$

where

$$\alpha_0 = \frac{8(3-2\beta)|\beta^3-3\beta^2+8\beta-12|-9(4-3\beta)(2-\beta)^2}{8(3-2\beta)|\beta^3-3\beta^2+8\beta-12|} + \sqrt{\frac{81(4-3\beta)^2(2-\beta)^4+24(2-\beta)^3|\beta^3-3\beta^2+8\beta-12|(8(3-2\beta)^2-3(4-3\beta)(2-\beta))}{8(3-2\beta)|\beta^3-3\beta^2+8\beta-12|}},$$

$$H(t_0) = \frac{(1-\alpha)^2}{(3-2\beta)^2} - \frac{c^2(\alpha, \beta)}{576(4-3\beta)(3-2\beta)^2(2-\beta)^4\Delta(\alpha, \beta)}, t_0 = \sqrt{\frac{-c(\alpha, \beta)}{\Delta(\alpha, \beta)}},$$

$$c(\alpha, \beta) = 144(2-\beta)^2(1-\alpha)^2 \{ (3-2\beta)(4-3\beta)(1-\alpha) + (2-\beta)(2(3-2\beta)^2 - (4-3\beta)(2-\beta)) \},$$

$$\Delta(\alpha, \beta) = 4(1-\alpha)^2 \{ 8(3-2\beta)^2|\beta^3-3\beta^2+8\beta-12|(1-\alpha)^2 - 18(4-3\beta)(3-2\beta)(2-\beta)^2(1-\alpha) + 3(2-\beta)^3(3(4-3\beta)(2-\beta) - 8(3-2\beta)^2) \}.$$

Proof. The proof of Corollary 3.4 is similar to the proof of Theorem 2.1. Therefore, the details of the proof of Corollary 3.4 may be omitted. □

Remark 3.5. Numerous consequences of the results obtained in this study can be deduced by choosing suitable values of the parameters involved. Many of these consequences were proved by recently studies on the subject (see, for example, [2]).

For example, by taking $\beta = 1$ and $\beta = 0$ in Corollary 3.4, we obtain the results for the classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$, which improves to the results obtained in Theorem 2.1 and 2.3 of [2], respectively. The results obtained in Theorem 2.1 and 2.3 of [2] can also be obtained by choosing $\phi(z)$ of the form (3.2) in Theorem 2.6 and 2.7.

By taking $\beta = 1, \beta = 0,$ and $\alpha = 0$ in Corollary 3.4, we get the boundary estimates for the second Hankel determinant in the classes of bi-starlike and bi-convex functions as $|a_2a_4 - a_3^2| \leq \frac{20}{3}$ and $|a_2a_4 - a_3^2| \leq \frac{1}{3}$. These consequences verifies to Corollary 2.2 and 2.4 of [2], respectively. The results

obtained in Corollary 2.2 and 2.4 of [2] can also be obtained by choosing $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$ in Theorem 2.6 and 2.7.

Thus, we can say that the results obtained in this paper are improvement of the results available in this area.

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