



Certain subclasses of p -valent functions defined by multiplier transformations

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Abstract

In the present paper, making use of the operator $\mathcal{D}_p^\delta(\lambda, \mu, l)$, two new subclasses $\mathcal{S}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$ and $\mathcal{ST}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$ of p -valent functions are investigated in the open unit disk. Some interesting relations and characteristics such as inclusion relationships, neighborhoods, partial sums and some applications of fractional calculus of functions belonging to each of these subclasses $\mathcal{S}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$ and $\mathcal{ST}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$ are investigated. Relevant connections of the definitions and results presented in this paper with those obtained in several earlier works on the subject are also pointed out.

Keywords Multiplier transformations · Analytic functions · p -valent functions · Neighborhoods and Partial sums of analytic functions · Fractional calculus operators

Mathematics Subject Classification Primary 30C45

1 Introduction and definitions

Let $\mathcal{A}(n, p)$ denote the class of functions normalized by

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}) \quad (1.1)$$

which are analytic and p -valent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

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Let $f(z)$ and $g(z)$ be analytic in \mathcal{U} . Then, we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in \mathcal{U} with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = g(w(z))$ ($z \in \mathcal{U}$). We denote this subordination $f \prec g$ or $f(z) \prec g(z)$ ($z \in \mathcal{U}$). In particular, if the function g is univalent in \mathcal{U} , the above subordination is equivalent to $f(0) = g(0)$, $f(\mathcal{U}) \subset g(\mathcal{U})$. For $f \in \mathcal{A}(n, p)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}) \tag{1.2}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) := z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k =: (g * f)(z) \quad (z \in \mathcal{U}). \tag{1.3}$$

Note that $f * g \in \mathcal{A}(n, p)$. In particular, we set

$$\mathcal{A}(p, 1) := \mathcal{A}_p, \quad \mathcal{A}(1, n) := \mathcal{A}(n), \quad \mathcal{A}(1, 1) := \mathcal{A}_1 = \mathcal{A}.$$

For $f \in \mathcal{A}(n, p)$ given by (1.1), Deniz and Orhan [13] defined the multiplier transformations $\mathcal{D}_p^\delta(\lambda, \mu, l)$ as follows:

Definition 1 Let $f \in \mathcal{A}(n, p)$. For the parameters $\delta, \lambda, \mu, l \in \mathbb{R}; \lambda \geq \mu \geq 0$ and $\delta, l \geq 0$ define the multiplier transformations $\mathcal{D}_p^\delta(\lambda, \mu, l)$ on $\mathcal{A}(n, p)$ by the following

$$\begin{aligned} \mathcal{D}_p^0(\lambda, \mu, l)f(z) &= f(z) \\ (p+l)\mathcal{D}_p^1(\lambda, \mu, l)f(z) &= \lambda\mu z^2 f''(z) + (\lambda - \mu + (1-p)\lambda\mu) z f'(z) + (p(1-\lambda + \mu) + l) f(z) \\ (p+l)\mathcal{D}_p^2(\lambda, \mu, l)f(z) &= \lambda\mu z^2 [\mathcal{D}_p^1(\lambda, \mu, l)f(z)]'' + (\lambda - \mu + (1-p)\lambda\mu) z [\mathcal{D}_p^1(\lambda, \mu, l)f(z)]' \\ &\quad + (p(1-\lambda + \mu) + l) \mathcal{D}_p^1(\lambda, \mu, l)f(z) \\ \mathcal{D}_p^{\delta_1}(\lambda, \mu, l)(\mathcal{D}_p^{\delta_2}(\lambda, \mu, l)f(z)) &= \mathcal{D}_p^{\delta_2}(\lambda, \mu, l)(\mathcal{D}_p^{\delta_1}(\lambda, \mu, l)f(z)) \end{aligned} \tag{1.4}$$

for $z \in \mathcal{U}$ and $p, n \in \mathbb{N} := \{1, 2, \dots\}$.

If f is given by (1.1) then from the definition of the multiplier transformations $\mathcal{D}_p^\delta(\lambda, \mu, l)$, we can easily see that

$$\mathcal{D}_p^\delta(\lambda, \mu, l)f(z) = z^p + \sum_{k=n+p}^{\infty} \Phi_p^k(\delta, \lambda, \mu, l) a_k z^k \tag{1.5}$$

where

$$\Phi_p^k(\delta, \lambda, \mu, l) = \left[\frac{(k-p)(\lambda\mu k + \lambda - \mu) + p + l}{p + l} \right]^\delta. \tag{1.6}$$

It should be remarked that the operator $\mathcal{D}_p^\delta(\lambda, \mu, l)$ is a generalization of many other operators considered earlier. In particular, for $f \in \mathcal{A}(n, p)$ we have the following:

- (1) $\mathcal{D}_1^\delta(1, 0, 0)f(z) \equiv D^\delta f(z)$, ($\delta \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) the Sălăgean differential operator [26].

- (2) $\mathcal{D}_1^\delta(\lambda, 0, 0)f(z) \equiv D_\lambda^\delta f(z)$, ($\delta \in \mathbb{N}_0$) the generalized Sălăgean differential operator introduced by Al-Oboudi [2].
- (3) $\mathcal{D}_1^\delta(\lambda, \mu, 0)f(z) \equiv D_{\lambda,\mu}^\delta f(z)$, the operator studied by Deniz and Orhan [12], in special case $0 \leq \mu \leq \lambda \leq 1$ the operator was studied firstly Răducanu and Orhan [24].
- (4) $\mathcal{D}_1^\delta(1, 0, l)f(z) \equiv I_l^\delta f(z)$, ($\delta \in \mathbb{N}_0$) the operator considered by Cho and Srivastava [9] and Cho and Kim [10].
- (5) $\mathcal{D}_1^\delta(1, 0, 1)f(z) \equiv I^\delta f(z)$, ($\delta \in \mathbb{N}_0$) the operator investigated by Uralegaddi and Somonatha [36].
- (6) $\mathcal{D}_1^\delta(\lambda, 0, 0)f(z) \equiv D_\lambda^\delta f(z)$, ($\delta \in \mathbb{R}^+ \cup \{0\}$) the operator studied by Acu and Owa [1].
- (7) $\mathcal{D}_1^\delta(\lambda, 0, l)f(z) \equiv I(\delta, \lambda, l)f(z)$, ($\delta \in \mathbb{R}^+ \cup \{0\}$) the operator introduced by Cataş [6].
- (8) $\mathcal{D}_p^\delta(1, 0, 0)f(z) \equiv D_p^\delta f(z)$, ($\delta \in \mathbb{N}_0$) the operator considered by Shenan et al. [27].
- (9) $\mathcal{D}_p^\delta(\lambda, 0, 0)f(z) \equiv D_{\lambda,p}^\delta f(z)$, ($\delta \in \mathbb{N}_0$) the operator investigated by Kwon [16].
- (10) $\mathcal{D}_p^\delta(1, 0, l)f(z) \equiv I_p(\delta, l)f(z)$, the operator considered by Kumar et al. [30].
- (11) $\mathcal{D}_p^\delta(\lambda, 0, l)f(z) \equiv I_p(\delta, \lambda, l)f(z)$, the operator studied recently by Cataş et al. [7].

For special values of parameters λ, μ, l and p , from the operator $\mathcal{D}_p^\delta(\lambda, \mu, l)$ the following new operators can be obtained:

- $\mathcal{D}_p^\delta(\lambda, \mu, 1) \equiv \mathcal{D}_p^\delta(\lambda, \mu)$
- $\mathcal{D}_1^\delta(\lambda, \mu, l) \equiv \mathcal{D}^\delta(\lambda, \mu, l)$.

Now, by making use of the operator $\mathcal{D}_p^\delta(\lambda, \mu, l)$, we define a new subclass of functions belonging to the class $\mathcal{A}(n, p)$.

Definition 2 Let $\lambda \geq \mu \geq 0; l, \delta \geq 0; p \in \mathbb{N}$ and for the parametres σ, A and B such that

$$-1 \leq A < B \leq 1, \text{ and } 0 \leq \sigma < p,$$

we say that a function $f(z) \in \mathcal{A}(n, p)$ is in the class $\mathcal{S}_{\lambda,\mu,l}^\delta(A, B; \sigma, p)$ if it satisfies the following subordination condition:

$$\frac{1}{p - \sigma} \left(\frac{z[\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)]'}{\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)} - \sigma \right) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}). \tag{1.7}$$

If the following inequality holds true,

$$\left| \frac{\frac{z[\mathcal{D}_p^\delta(\lambda,\mu,l)f(z)]'}{\mathcal{D}_p^\delta(\lambda,\mu,l)f(z)} - p}{B \frac{z[\mathcal{D}_p^\delta(\lambda,\mu,l)f(z)]'}{\mathcal{D}_p^\delta(\lambda,\mu,l)f(z)} - [pB + (A - B)(p - \sigma)]} \right| < 1 \quad (z \in \mathcal{U}) \tag{1.8}$$

the inequality (1.5) is equivalent the subordination condition (1.6).

We note that by specializing the parameters $\lambda, \mu, l, \delta, \sigma, A, B$ and p , $\mathcal{S}_{\lambda,\mu,l}^\delta(A, B; \sigma, p)$ reduces to the following well-known subclasses of analytic functions:

- (i) $\mathcal{S}_{1,0,0}^0(-1, 1; \sigma, 1) = \mathcal{S}^*(\sigma)$ (Silverman [28]);
- (ii) $\mathcal{S}_{1,0,0}^0(A, B; 0, p) = T^*(A, B)$ (Goel and Sohi [14]).

Also, by making use of the operator $\mathcal{D}_p^\delta(\lambda, \mu, l)$, Çağlar et al. [5] defined a different subclass of $\mathcal{A}(n, p)$.

Furthermore, we say that a function $f(z) \in \mathcal{ST}_{\lambda,\mu,l}^\delta(A, B; \sigma, p)$ is in the subclass $\mathcal{S}_{\lambda,\mu,l}^\delta(A, B; \sigma, p)$ if $f(z)$ is of the following form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} |a_k| z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}). \tag{1.9}$$

In our present paper, we shall make use of the familiar *integral operator* $\mathcal{I}_{\vartheta,p}$ defined by (see, for details, [4, 17, 19, 36])

$$(\mathcal{I}_{\vartheta,p})(z) := \frac{\vartheta + p}{z^p} \int_0^z t^{\vartheta-1} f(t)dt \quad (f \in \mathcal{A}(n, p); \vartheta + p > 0; p \in \mathbb{N}) \tag{1.10}$$

as well as the fractional calculus operator \mathfrak{D}_z^ν for which it is well known that (see, for details, [22, 31] and [32, 33])

$$\mathfrak{D}_z^\nu \{z^\rho\} = \frac{\Gamma(\rho + 1)}{\Gamma(\rho + 1 - \nu)} z^{\rho-\nu} \quad (\rho > -1; \nu \in \mathbb{R}) \tag{1.11}$$

in terms of Gamma function.

The main object of the present paper is to investigate the various important properties and characteristics of two subclasses of $\mathcal{A}(n, p)$ of normalized analytic functions in \mathcal{U} with negative and positive coefficients, which are introduced here by making use of the multiplier transformations $\mathcal{D}_p^\delta(\lambda, \mu, l)$ defined by (1.4). Inclusion relationships for the class $\mathcal{S}_{\lambda,\mu,l}^\delta(A, B; \sigma, p)$ are investigated by applying the techniques of convolution. Furthermore, several properties involving generalized neighborhoods and partial sums for functions belonging to these subclasses are investigated. Finally, some applications of fractional calculus operators are considered. Relevant connections of the definitions and results presented here with those obtained in several earlier works are also pointed out.

2 Basic properties of the function class $\mathcal{ST}_{\lambda,\mu,l}^\delta(A, B; \sigma, p)$

We first determine a necessary and sufficient condition for a function $f(z) \in \mathcal{A}(n, p)$ of the form (1.4) to be in the class $\mathcal{ST}_{\lambda,\mu,l}^\delta(A, B; \sigma, p)$.

Theorem 1 *Let the function $f(z) \in \mathcal{A}(n, p)$ be defined by (1.9). Then the function $f(z)$ is in the class $\mathcal{ST}_{\lambda,\mu,l}^\delta(A, B; \sigma, p)$ if and only if*

$$\sum_{k=n+p}^\infty [(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l) |a_k| \leq (B - A)(p - \sigma) \tag{2.1}$$

where $\Phi_p^k(\delta, \lambda, \mu, l)$ is given (1.6).

Proof If the condition (2.1) hold true, we find from (1.8) and (2.1) that

$$\begin{aligned} & \left| z[\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)]' - p\mathcal{D}_p^\delta(\lambda, \mu, l)f(z) \right| \\ &= \left| Bz[\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)]' - [pB + (A - B)(p - \sigma)]\mathcal{D}_p^\delta(\lambda, \mu, l) \right| \\ &= \left| - \sum_{k=n+p}^\infty (k - p) \Phi_p^k(\delta, \lambda, \mu, l) |a_k| z^k \right| \\ &= \left| - \left[(A - B)(p - \sigma)z^p + \sum_{k=n+p}^\infty (B(k - p) + (B - A)(p - \sigma)) \Phi_p^k(\delta, \lambda, \mu, l) |a_k| z^k \right] \right| \\ &\leq \sum_{k=n+p}^\infty [(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l) |a_k| \\ & \quad - (B - A)(p - \sigma) \leq 0 \quad (z \in \partial\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}). \end{aligned}$$

Hence, by the *Maximum Modulus Theorem*, we have

$$f(z) \in ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p).$$

Conversely, assume that the function $f(z)$ defined by (2.7) is in the class $ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$. Then we have

$$\begin{aligned} & \left| \frac{\frac{z[\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)]'}{\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)} - p}{B \frac{z[\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)]'}{\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)} - [pB + (A - B)(p - \sigma)]} \right| \\ &= \left| \frac{\sum_{k=n+p}^\infty (k - p) \Phi_p^k(\delta, \lambda, \mu, l) |a_k| z^k}{\sum_{k=n+p}^\infty [-B(k - p) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l) |a_k| z^k + (B - A)(p - \sigma) z^p} \right| \\ &< 1 \quad (z \in \mathcal{U}). \end{aligned} \tag{2.2}$$

Now, since $|\Re(z)| \leq |z|$ for all z , we have

$$\Re \left(\frac{\sum_{k=n+p}^\infty (k - p) \Phi_p^k(\delta, \lambda, \mu, l) |a_k| z^{k-p}}{\sum_{k=n+p}^\infty [-B(k - p) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l) |a_k| z^{k-p} + (B - A)(p - \sigma) z^p} \right) < 1. \tag{2.3}$$

We choose values of z on the real axis so that the following expression:

$$\frac{z[\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)]'}{\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)}$$

is real. Then, upon clearing the denominator in (2.3) and letting $z \rightarrow 1^-$ through real values, we get the following inequality

$$\sum_{k=n+p}^\infty [(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l) |a_k| \leq (B - A)(p - \sigma).$$

This completes the proof of Theorem 1. □

Corollary 1 *Let the function $f(z) \in \mathcal{A}(n, p)$ be defined by (1.2). If the function $f(z)$ is in the class $\mathcal{S}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$, then*

$$\sum_{k=n+p}^\infty [(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l) |a_k| \leq (B - A)(p - \sigma). \tag{2.4}$$

Corollary 2 *Let the function $f(z) \in \mathcal{A}(n, p)$ be defined by (1.9). If the function $f(z) \in ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$, then*

$$|a_k| \leq \frac{(B - A)(p - \sigma)}{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)} \quad (k, p \in \mathbb{N}). \tag{2.5}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B - A)(p - \sigma)}{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)} z^k \quad (k, p \in \mathbb{N}). \tag{2.6}$$

We next prove the following growth and distortion properties for the class $ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$.

Theorem 2 *If a function $f(z)$ be defined by (1.2) is in the class $ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$, then*

$$\begin{aligned} & \left(\frac{p!}{(p-q)!} - \frac{(B-A)(p-\sigma)(n+p)!}{(n+p-q)! [n(1+B) + (B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right) |z|^{p-q} \\ & \leq |f^{(q)}(z)| \leq \left(\frac{p!}{(p-q)!} \right. \\ & \quad \left. + \frac{(B-A)(p-\sigma)(n+p)!}{(n+p-q)! [n(1+B) + (B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right) |z|^{p-q} \quad (2.7) \end{aligned}$$

for $q \in \mathbb{N}_0$, $p > q$ and all $z \in \mathcal{U}$. The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)(p-\sigma)}{(n+p) [n(1+B) + (B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} z^{n+p} \quad (p \in \mathbb{N}). \quad (2.8)$$

Proof In view of Theorem 1 we have

$$\sum_{k=n+p}^\infty \frac{[(k-p)(1+B) + (B-A)(p-\sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |a_k| \leq 1,$$

which readily yields we obtain

$$\frac{(n+p-q)! [n(1+B) + (B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)(n+p)!} \sum_{k=n+p}^\infty \frac{k!}{(k-q)!} |a_k| \leq 1,$$

or

$$\sum_{k=n+p}^\infty \frac{k!}{(k-q)!} |a_k| \leq \frac{(B-A)(p-\sigma)(n+p)!}{(n+p-q)! [n(1+B) + (B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)}. \quad (2.9)$$

From last inequality,

$$\begin{aligned} |f^{(q)}(z)| &= \left| \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=n+p}^\infty \frac{k!}{(k-q)!} |a_k| z^{k-q} \right| \quad (q \in \mathbb{N}_0; p > q) \\ &\leq \frac{p!}{(p-q)!} + \frac{(B-A)(p-\sigma)(n+p)!}{(n+p-q)! [n(1+B) + (B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)}. \end{aligned} \quad (2.10)$$

Similarly

$$\begin{aligned} |f^{(q)}(z)| &= \left| \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=n+p}^\infty \frac{k!}{(k-q)!} |a_k| z^{k-q} \right| \quad (q \in \mathbb{N}_0; p > q) \\ &\geq \frac{p!}{(p-q)!} - \frac{(B-A)(p-\sigma)(n+p)!}{(n+p-q)! [n(1+B) + (B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)}. \end{aligned} \quad (2.11)$$

This completes the proof of Theorem 2. □

For $q = 0$ in Theorem 2, we get obtain the following distortion result for $ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$.

Corollary 3 Let $f(z) \in ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$. Then we have

$$\begin{aligned} & \left(1 - \frac{(B - A)(p - \sigma)}{[n(1 + B) + (B - A)(p - \sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right) |z|^p \\ & \leq |f(z)| \leq \left(1 + \frac{(B - A)(p - \sigma)}{[n(1 + B) + (B - A)(p - \sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right) |z|^p \end{aligned}$$

In Theorem 2 for $q = 1$, we get obtain the following growth result for $ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$.

Corollary 4 Let $f(z) \in ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$. Then we have

$$\begin{aligned} & \left(p - \frac{(B - A)(p - \sigma)(n + p)}{[n(1 + B) + (B - A)(p - \sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right) |z|^{p-1} \\ & \leq |f'(z)| \leq \left(p + \frac{(B - A)(p - \sigma)(n + p)}{[n(1 + B) + (B - A)(p - \sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right) |z|^{p-1} \end{aligned}$$

Theorem 3 Let $f_{n+p-1}(z) = z^p$ and

$$f_k(z) = z^p - \frac{(B - A)(p - \sigma)}{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)} z^k.$$

Then $f(z) \in ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$ if and only if it can be expressed in the form $f(z) = \sum_{k=n+p-1}^\infty \gamma_k f_k(z)$ where $\gamma_k > 0$ and $\sum_{k=n+p-1}^\infty \gamma_k = 1$.

Proof Suppose

$$f(z) = \sum_{k=n+p-1}^\infty \gamma_k f_k(z).$$

Then we have

$$\begin{aligned} f(z) &= \sum_{k=n+p-1}^\infty \gamma_k f_k(z) = \gamma_{n+p-1} f_{n+p-1}(z) + \sum_{k=n+p}^\infty \gamma_k f_k(z) \\ &= \left(1 - \sum_{k=n+p}^\infty \gamma_k \right) z^p + \sum_{k=n+p}^\infty \gamma_k f_k(z) \\ &= z^p - \sum_{k=n+p}^\infty \gamma_k \left[-z^p + \frac{(B - A)(p - \sigma)}{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)} z^k + z^p \right] \\ &= z^p - \sum_{k=n+p}^\infty \gamma_k \frac{(B - A)(p - \sigma)}{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)} z^k \end{aligned}$$

Thus, $f(z) \in ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$.

Converselly, suppose $f(z) \in ST_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$. Since

$$|a_k| \leq \frac{(B - A)(p - \sigma)}{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)} \quad (k = n + p, n + p + 1, \dots),$$

We may set

$$\gamma_k = z^p - \frac{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} |a_k|, \\ (k = n + p, n + p + 1, \dots),$$

and

$$\gamma_{n+p-1} = 1 - \sum_{k=n+p-1}^{\infty} \gamma_k f_k(z).$$

Then $f(z) = \sum_{k=n+p-1}^{\infty} \gamma_k f_k(z)$.

This complete the proof. □

Corollary 5 *The extreme points of $ST_{\lambda, \mu, l}^{\delta}(A, B; \sigma, p)$ is*

$$f_k(z) = z^p - \frac{(B - A)(p - \sigma)}{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)} z^k, \\ (k = n + p, n + p + 1, \dots).$$

Theorem 4 *If $f \in ST_{\lambda, \mu, l}^{\delta}(A, B; \sigma, p)$, then f is starlike of order α in the disk*

$$|z| \leq r_1 = \inf \left\{ \frac{(p - \alpha)[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(k - \alpha)(B - A)(p - \sigma)} \right\}^{\frac{1}{k-p}},$$

where $0 \leq \alpha < p$ and $p < k$.

Proof For $0 \leq \alpha < p$ we need to show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \alpha,$$

that is,

$$\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{\sum_{k=n+p}^{\infty} (p - k) |a_k| z^k}{z^p - \sum_{k=n+p}^{\infty} |a_k| z^k} \right| \\ \leq \frac{\sum_{k=n+p}^{\infty} (k - p) |a_k| |z|^{k-p}}{1 - \sum_{k=n+p}^{\infty} |a_k| |z|^{k-p}} < p - \alpha$$

or

$$\sum_{k=n+p}^{\infty} \left(\frac{k - \alpha}{p - \alpha} \right) |a_k| |z|^{k-p} \leq 1.$$

By using Theorem 1, the above inequality holds if

$$|z|^{k-p} \leq \frac{(p - \alpha)[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(k - \alpha)(B - A)(p - \sigma)}.$$

This completes the proof of theorem. □

We now determine the radius of convexity for functions in $ST_{\lambda, \mu, l}^{\delta}(A, B; \sigma, p)$.

Theorem 5 If $f \in \mathcal{ST}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$, then f is convex of order α in the disk

$$|z| \leq r_2 = \inf \left\{ \frac{p(p - \alpha) [(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{k(k - \alpha)(B - A)(p - \sigma)} \right\}^{\frac{1}{k-p}},$$

where $0 \leq \alpha < p$ and $p < k$.

Proof It suffices to show that

$$\left| \frac{zf'(z)}{f'(z)} + 1 - p \right| < p - \alpha$$

for $|z| \leq r_2$. Then proof is similar to the proof of Theorem 4 and therefore we omit the details. \square

3 Inclusion relations involving neighborhoods

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by [15], [25] and others including Srivastava et al. [31–34], [20, 21], Deniz et al. [11], Aouf et al. [3] (see also [6]).

Firstly, we define the (n, τ) -neighborhood of function $f(z) \in \mathcal{A}(n, p)$ of the form (1.1) by means of Definition 3 below.

Definition 3 For $\tau > 0$ and a non-negative sequence $\mathcal{S} = \{s_k\}_{k=1}^\infty$, where

$$s_k := \frac{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} \quad (k \in \mathbb{N}). \tag{3.1}$$

The (n, τ) -neighborhood of a function $f(z) \in \mathcal{A}(n, p)$ of the form (1.1) is defined as follows:

$$\mathcal{N}_{n,p}^\tau(f) := \left\{ g : g(z) = z^p + \sum_{k=n+p}^\infty b_k z^k \in \mathcal{A}(n, p) \text{ and } \sum_{k=n+p}^\infty s_k |b_k - a_k| \leq \tau \ (\tau > 0) \right\}. \tag{3.2}$$

For $s_k = k$, Definition 3 would correspond to the \mathcal{N}_τ -neighborhood considered by [25].

Our first result based upon the familiar concept of neighborhood defined by (3.1).

Theorem 6 Let $f(z) \in \mathcal{S}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$ be given by (1.1). If f satisfies the inclusion condition:

$$(f(z) + \varepsilon z^p)(1 + \varepsilon)^{-1} \in \mathcal{S}_{\lambda, \mu, l}^\delta(A, B; \sigma, p) \quad (\varepsilon \in \mathbb{C}; |\varepsilon| < \tau; \tau > 0), \tag{3.3}$$

then

$$\mathcal{N}_{n,p}^\tau(f) \subset \mathcal{S}_{\lambda, \mu, l}^\delta(A, B; \sigma, p). \tag{3.4}$$

Proof It is not difficult to see that a function f belongs to $\mathcal{S}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$ if and only if

$$\frac{\frac{z[\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)]'}{\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)} - p}{B \frac{z[\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)]'}{\mathcal{D}_p^\delta(\lambda, \mu, l)f(z)} - [pB + (A - B)(p - \sigma)]} \neq \varpi \quad (z \in \mathcal{U}; \varpi \in \mathbb{C}, |\varpi| = 1), \tag{3.5}$$

which is equivalent to

$$(f * h)(z)/z^p \neq 0 \quad (z \in \mathcal{U}), \tag{3.6}$$

where for convenience,

$$h(z) := z^p + \sum_{k=n+p}^{\infty} c_k z^k = z^p + \frac{[(k-1) - (kB - (B-A)(p-\sigma))\varpi] \Phi_p^k(\delta, \lambda, \mu, l)}{\varpi(B-A)(p-\sigma)} z^k. \tag{3.7}$$

We easily find from (3.7) that

$$\begin{aligned} |c_k| &\leq \left| \frac{[(k-1) - (kB - (B-A)(p-\sigma))\varpi] \Phi_p^k(\delta, \lambda, \mu, l)}{\varpi(B-A)(p-\sigma)} \right| \\ &\leq \left| \frac{[(k-1) - (kB - (B-A)(p-\sigma))] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} \right| \quad (k \in \mathbb{N}). \end{aligned} \tag{3.8}$$

Furthermore, under the hypotheses of theorem, (3.3) and (3.6) yields the following inequalities:

$$\frac{((f(z) + \varepsilon z^p)(1 + \varepsilon)^{-1}) * h(z)}{z^p} \neq 0 \quad (z \in \mathcal{U})$$

or

$$\frac{f(z) * h(z)}{z^p} \neq \varepsilon \quad (z \in \mathcal{U}),$$

which is equivalent to the following:

$$\frac{f(z) * h(z)}{z^p} \geq \tau \quad (z \in \mathcal{U}; \tau > 0). \tag{3.9}$$

Now, if we let

$$g(z) := z^p + \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{N}_{n,p}^{\tau}(f),$$

then we have

$$\begin{aligned} \left| \frac{(f(z) - g(z)) * h(z)}{z^p} \right| &= \left| \sum_{k=n+p}^{\infty} (a_k - b_k) c_k z^{k-p} \right| \\ &\leq \sum_{k=n+p}^{\infty} \frac{[(k-1) - (kB - (B-A)(p-\sigma))] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |a_k - b_k| |z|^{k-p} < \tau \\ &\quad (z \in \mathcal{U}; \tau > 0). \end{aligned}$$

Thus, for any complex number ϖ such that $|\varpi| = 1$, we have

$$(g * h)(z)/z^p \neq 0 \quad (z \in \mathcal{U}),$$

which implies that $g \in \mathcal{S}_{\lambda, \mu, l}^{\delta}(A, B; \sigma, p)$. The proof is complete. □

We now define the (n, τ) -neighborhood of a function $f(z) \in \mathcal{T}(n, p)$ of the form (1.9) as follows:

Definition 4 The (n, τ) -neighborhood of a function $f(z) = \sum_{k=n+p}^{\infty} |a_k| z^k$ of the form (1.9) is given by

$$\tilde{N}_{n,p}^{\tau}(f) := \left\{ g : g(z) = z^p - \sum_{k=n+p}^{\infty} |b_k| z^k \text{ and } \sum_{k=n+p}^{\infty} \frac{[(k-p)(1+B) + (B-A)(p-\sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} \||b_k| - |a_k|\| \leq \tau \right\}. \tag{3.10}$$

Theorem 7 If the function $f(z)$ defined by (1.9) is in the class $ST_{\lambda,\mu,l}^{\delta+1}(A, B; \sigma, p)$, then

$$\tilde{N}_{n,p}^{\tau}(f) \subset ST_{\lambda,\mu,l}^{\delta}(A, B; \sigma, p) \tag{3.11}$$

where

$$\tau = \frac{n[\lambda\mu(n+p) + \lambda - \mu]}{n[\lambda\mu(n+p) + \lambda - \mu] + p + l}.$$

The result is the best possible in the sense that τ cannot be increased.

Proof For a function $f(z) \in ST_{\lambda,\mu,l}^{\delta+1}(A, B; \sigma, p)$ of the form (1.9) Theorem 1 immediately yields

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(1+B) + (B-A)(p-\sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |a_k| \tag{3.12}$$

$$\leq \frac{p+l}{n[\lambda\mu(n+p) + \lambda - \mu] + p + l}. \tag{3.13}$$

Similarly, by taking

$$g(z) = z^p - \sum_{k=n+p}^{\infty} |b_k| z^k \in \tilde{N}_{n,p}^{\tau}(f) \left(\tau = \frac{n[\lambda\mu(n+p) + \lambda - \mu]}{n[\lambda\mu(n+p) + \lambda - \mu] + p + l} \right),$$

we find from the definition Definition 4 that

$$\sum_{k=n+p}^{\infty} \frac{[(k-p)(1+B) + (B-A)(p-\sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} \||b_k| - |a_k|\| \leq \tau \quad (\tau > 0). \tag{3.14}$$

With the help of (3.12) and (3.14), we have

$$\begin{aligned} & \sum_{k=n+p}^{\infty} \frac{[(k-p)(1+B) + (B-A)(p-\sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |b_k| \\ & \leq \sum_{k=n+p}^{\infty} \frac{[(k-p)(1+B) + (B-A)(p-\sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |a_k| \\ & + \sum_{k=n+p}^{\infty} \frac{[(k-p)(1+B) + (B-A)(p-\sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} \||b_k| - |a_k|\| \\ & \leq \frac{p+l}{n[\lambda\mu(n+p) + \lambda - \mu] + p + l} + \tau = 1. \end{aligned}$$

Hence, in view of the Theorem 1 again, we see that $g(z) \in ST_{\lambda, \mu, l}^{\delta+1}(A, B; \sigma, p)$.

To show the sharpness of the assertion of Theorem 7, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^p - \left[\frac{(B - A)(p - \sigma)}{[n(1 + B) + (B - A)(p - \sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} \right] z^{n+p} \in ST_{\lambda, \mu, l}^{\delta+1}(A, B; \sigma, p) \tag{3.15}$$

and

$$g(z) = z^p - \left[\frac{(B - A)(p - \sigma)}{[n(1 + B) + (B - A)(p - \sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} + \frac{(B - A)(p - \sigma)}{[n(1 + B) + (B - A)(p - \sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} \tau^* \right] z^{n+p} \tag{3.16}$$

where $\tau^* > \tau$. Clearly, the function $g(z)$ belong to $\tilde{N}_{n,p}^{\eta^*}(f)$. On the other hand, we find from Theorem 1 that $g(z) \notin ST_{\lambda, \mu, l}^{\delta+1}(A, B; \sigma, p)$. This evidently completes the proof of Theorem 7. \square

4 Partial sums of the function class $ST_{\lambda, \mu, l}^{\delta+1}(A, B; \sigma, p)$

Following the earlier work by [29] and recently [18] and Deniz et al. [11], in this section we investigate the ratio of real parts of functions involving (1.9) and its sequence of partial sums defined by

$$\kappa_m(z) = \begin{cases} z^p, & m = 1, 2, \dots, n + p - 1; \\ z^p - \sum_{k=n+p}^m |a_k| z^k, & m = n + p, n + p + 1, \dots \end{cases} \quad (k \geq n + p; n, p \in \mathbb{N}) \tag{4.1}$$

and determine sharp lower bounds for $\Re \{f(z)/\kappa_m(z)\}$, $\Re \{\kappa_m(z)/f(z)\}$.

Theorem 8 *Let $f \in \mathcal{A}(n, p)$ and $\kappa_m(z)$ be given by (1.9) and (4.1), respectively. Suppose also that*

$$\sum_{k=n+p}^{\infty} \lambda_k |a_k| \leq 1 \quad \left(\lambda_k = \frac{[(k - p)(1 + B) + (B - A)(p - \sigma)] \Phi_p^k(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} \right). \tag{4.2}$$

Then for $m \geq k + p$, we have

$$\Re \left(\frac{f(z)}{\kappa_m(z)} \right) > 1 - \frac{1}{\lambda_{m+1}} \tag{4.3}$$

and

$$\Re \left(\frac{\kappa_m(z)}{f(z)} \right) > \frac{\lambda_{m+1}}{1 + \lambda_{m+1}}. \tag{4.4}$$

The results are sharp for every m with the extremal functions given by

$$f(z) = z^p - \frac{1}{\lambda_{m+1}} z^{m+1}. \tag{4.5}$$

Proof Under the hypothesis of the theorem, we can see from (4.2) that

$$\lambda_{k+1} > \lambda_k > 1 \quad (k \geq n + p).$$

Therefore, we have

$$\sum_{k=n+p}^m |a_k| + \lambda_{m+1} \sum_{k=m+1}^{\infty} |a_k| \leq \sum_{k=n+p}^{\infty} \lambda_k |a_k| \leq 1 \tag{4.6}$$

by using hypothesis (4.2) again. Upon setting

$$\begin{aligned} \omega(z) &= \lambda_{m+1} \left[\frac{f(z)}{\kappa_m(z)} - \left(1 - \frac{1}{\lambda_{m+1}} \right) \right] \\ &= 1 - \frac{\lambda_{m+1} \sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{1 - \sum_{k=n+p}^m |a_k| z^{k-p}}. \end{aligned} \tag{4.7}$$

By applying (4.6) and (4.7), we find that

$$\begin{aligned} \left| \frac{\omega(z) - 1}{\omega(z) + 1} \right| &= \left| \frac{-\lambda_{m+1} \sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{2 - 2 \sum_{k=n+p}^m |a_k| z^{k-p} - \lambda_{m+1} \sum_{k=m+1}^{\infty} |a_k| z^{k-p}} \right| \\ &\leq \frac{\lambda_{m+1} \sum_{k=m+1}^{\infty} |a_k|}{2 - 2 \sum_{k=n+p}^m |a_k| - \lambda_{m+1} \sum_{k=m+1}^{\infty} |a_k|} \leq 1 \quad (z \in \mathcal{U}; k \geq n+p), \end{aligned} \tag{4.8}$$

which shows that $\Re(\omega(z)) > 0$ ($z \in \mathcal{U}$). From (4.7), we immediately obtain the inequality (4.3). To see that the function f given by (4.5) gives the sharp result, we observe for $z \rightarrow 1^-$ that

$$\begin{aligned} \frac{f(z)}{\kappa_m(z)} &= 1 - \frac{1}{\lambda_{m+1}} z^{m-p+1} \rightarrow 1 - \frac{1}{\lambda_{m+1}}, \\ \psi(z) &= (1 + \lambda_{m+1}) \left[\frac{\kappa_m(z)}{f(z)} - \frac{\lambda_{m+1}}{1 + \lambda_{m+1}} \right] \\ &= 1 + \frac{(1 + \lambda_{m+1}) \sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{1 - \sum_{k=n+p}^m |a_k| z^{k-p}}, \end{aligned} \tag{4.9}$$

which shows that the bound in (4.3) is the best possible. Similarly, if we put and make use of (4.6), we can deduce that

$$\begin{aligned} \left| \frac{\psi(z) - 1}{\psi(z) + 1} \right| &= \left| \frac{(1 + \lambda_{m+1}) \sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{2 - 2 \sum_{k=n+p}^m |a_k| z^{k-p} + (\lambda_{m+1} - 1) \sum_{k=m+1}^{\infty} |a_k| z^{k-p}} \right| \\ &\leq \frac{(1 + \lambda_{m+1}) \sum_{k=m+1}^{\infty} |a_k|}{2 - 2 \sum_{k=n+p}^m |a_k| - (\lambda_{m+1} - 1) \sum_{k=m+1}^{\infty} |a_k|} \leq 1 \quad (z \in \mathcal{U}; k \geq n+p), \end{aligned} \tag{4.10}$$

which leads us immediately to assertion (4.4) of the theorem. The bound in (4.4) is sharp with the extremal function given by (4.5). The proof of theorem is thus completed. \square

5 Applications of fractional calculus operators

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (*cf.*, e.g., [23, 32, 33, 35]; see also [8, 32, 33] the various references cited therein). For our present investigation, we recall the following definitions.

Definition 5 Let $f(z)$ be analytic in a simply connected region of the z -plane containing the origin. The fractional integral of f of order ν is defined by

$$\mathfrak{D}_z^{-\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\nu}} d\zeta \quad (\nu > 0), \tag{5.1}$$

where the multiplicity of $(z - \zeta)^{\nu-1}$ is removed by requiring that $\log(z - \zeta)$ is real for $z - \zeta > 0$.

Definition 6 Let $f(z)$ be analytic in a simply connected region of the z -plane containing the origin. The fractional derivative of f of order ν is defined by

$$\mathcal{D}_z^\nu f(z) = \frac{1}{\Gamma(1 - \nu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^\nu} d\zeta \quad (0 \leq \nu < 1), \tag{5.2}$$

where the multiplicity of $(z - \zeta)^{-\nu}$ is removed by requiring that $\log(z - \zeta)$ is real for $z - \zeta > 0$.

Definition 7 Under the hypotheses of Definition 6, the fractional derivative of order $n + \nu$ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{n+\nu} f(z) = \frac{d^n}{dz^n} \{ \mathcal{D}_z^\nu f(z) \} \quad (0 \leq \nu < 1; n \in \mathbb{N}_0). \tag{5.3}$$

In this section, we shall investigate the growth and distortion properties of functions in the class $\mathcal{ST}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$, which involving the operators $\mathcal{I}_{\vartheta, p}$ and \mathcal{D}_z^ν . In order to derive our results, we need the following lemma given by Chen *et al.* [8].

Lemma 1 (see [8]). Let the function $f(z)$ defined by (1.9). Then

$$\begin{aligned} \mathcal{D}_z^\nu \{ (\mathcal{I}_{\vartheta, p} f)(z) \} &= \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \nu)} z^{p-\nu} - \sum_{k=n+p}^\infty \frac{(\vartheta + p)\Gamma(k + 1)}{(\vartheta + k)\Gamma(k + 1 - \nu)} a_k z^{k-\nu} \\ & \quad (\nu \in \mathbb{R}; \vartheta > -p; p, n \in \mathbb{N}) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \mathcal{I}_{\vartheta, p} \{ (\mathcal{D}_z^\nu f)(z) \} &= \frac{(\vartheta + p)\Gamma(p + 1)}{(\vartheta + p - \nu)\Gamma(p + 1 - \nu)} z^{p-\nu} - \sum_{k=n+p}^\infty \frac{(\vartheta + p)\Gamma(k + 1)}{(\vartheta + k - \nu)\Gamma(k + 1 - \nu)} a_k z^{k-\nu} \\ & \quad (\nu \in \mathbb{R}; \vartheta > -p; p, n \in \mathbb{N}) \end{aligned} \tag{5.5}$$

provided that no zeros appear in the denominators in (5.4) and (5.5).

Theorem 9 Let the functions $f(z)$ defined by (1.9) be in the class $\mathcal{ST}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)$. Then

$$\begin{aligned} |\mathcal{D}_z^{-\nu} \{ (\mathcal{I}_{\vartheta, p} f)(z) \}| &\geq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 + \nu)} \right. \\ & \quad \left. - \frac{(\vartheta + p)\Gamma(n + p + 1)(B - A)(p - \sigma)}{(\vartheta + n + p)\Gamma(n + p + 1 + \nu) [n(1 + B) + (B - A)(p - \sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right\} \\ & \quad |z|^{p+\nu} \quad (z \in \mathcal{U}; \nu > 0; \vartheta > -p; p, n \in \mathbb{N}) \end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
 |\mathcal{D}_z^{-\nu}\{(\mathcal{I}_{\vartheta,p}f)(z)\}| \leq & \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\nu)} \right. \\
 & + \frac{(\vartheta+p)\Gamma(n+p+1)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1+\nu)[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta,\lambda,\mu,l)} |z|^n \left. \right\} \\
 & |z|^{p+\nu} (z \in \mathcal{U}; \nu > 0; \vartheta > -p; p, n \in \mathbb{N}). \tag{5.7}
 \end{aligned}$$

Each of the assertions (5.6) and (5.7) is sharp.

Proof In view of Theorem 1, we have

$$\begin{aligned}
 & \frac{[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)} \sum_{k=n+p}^{\infty} |a_k| \\
 & \leq \sum_{k=n+p}^{\infty} \frac{[(k-p)(1+B)+(B-A)(p-\sigma)]\Phi_p^k(\delta,\lambda,\mu,l)|a_k|}{(B-A)(p-\sigma)} \leq 1, \tag{5.8}
 \end{aligned}$$

which readily yields

$$\sum_{k=n+p}^{\infty} |a_k| \leq \frac{(B-A)(p-\sigma)}{[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta,\lambda,\mu,l)}. \tag{5.9}$$

Consider the function $\mathcal{F}(z)$ defined in \mathcal{U} by

$$\begin{aligned}
 \mathcal{F}(z) &= \frac{\Gamma(p+1-\nu)}{\Gamma(p+1)} z^{-\nu} \mathcal{D}_z^{\nu}\{(\mathcal{I}_{\vartheta,p}f)(z)\} \\
 &= z^p - \sum_{k=n+p}^{\infty} \frac{(\vartheta+p)\Gamma(k+1)\Gamma(p+1+\nu)}{(\vartheta+k)\Gamma(k+1+\nu)\Gamma(p+1)} |a_k| z^k \\
 &= z^p - \sum_{k=n+p}^{\infty} \Theta(k) |a_k| z^k (z \in \mathcal{U})
 \end{aligned}$$

where

$$\Theta(k) := \frac{(\vartheta+p)\Gamma(k+1)\Gamma(p+1+\nu)}{(\vartheta+k)\Gamma(k+1+\nu)\Gamma(p+1)} (k \geq p+n; p, n \in \mathbb{N}; \nu > 0). \tag{5.10}$$

Since $\Theta(k)$ is a decreasing function of k when $\nu > 0$, we get

$$\begin{aligned}
 0 < \Theta(k) \leq \Theta(n+p) &= \frac{(\vartheta+p)\Gamma(n+p+1)\Gamma(p+1+\nu)}{(\vartheta+n+p)\Gamma(n+p+1+\nu)\Gamma(p+1)} \\
 & (\nu > 0; \vartheta > -p; p, n \in \mathbb{N}). \tag{5.11}
 \end{aligned}$$

Thus, by using (5.9) and (5.11), for all $z \in \mathcal{U}$, we deduce that

$$\begin{aligned}
 |\mathcal{F}(z)| \geq & |z|^p - \Theta(n+p) |z|^{n+p} \sum_{k=n+p}^{\infty} |a_k| \geq |z|^p \\
 & - \frac{(\vartheta+p)\Gamma(n+p+1)\Gamma(p+1+\nu)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1+\nu)\Gamma(p+1)[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta,\lambda,\mu,l)} |z|^{n+p}
 \end{aligned}$$

and

$$|\mathcal{F}(z)| \leq |z|^p + \Theta(n+p) |z|^{n+p} \sum_{k=n+p}^{\infty} |a_k| \leq |z|^p + \frac{(\vartheta+p)\Gamma(n+p+1)\Gamma(p+1+\nu)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1+\nu)\Gamma(p+1)[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^{n+p}$$

which yield the inequalities (5.6) and (5.7) of Theorem 9. Equalities in (5.6) and (5.7) are attained for the function $f(z)$ given by

$$\mathcal{D}_z^{-\nu}\{(\mathcal{I}_{\vartheta,p}f)(z)\} = \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\nu)} - \frac{(\vartheta+p)\Gamma(n+p+1)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1+\nu)[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta, \lambda, \mu, l)} z^n \right\} z^{p+\nu}$$

or, equivalently, by

$$(\mathcal{I}_{\vartheta,p}f)(z) = z^p - \frac{(\vartheta+p)(B-A)(p-\sigma)}{(\vartheta+n+p)[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta, \lambda, \mu, l)} z^{n+p}.$$

Thus, we complete the proof of theorem. □

Theorem 10 *Let the functions $f(z)$ defined by (1.9) be in the class $\mathcal{ST}_{\lambda,\mu,l}^{\delta}(A, B; \sigma, p)$. Then*

$$|\mathcal{D}_z^{\nu}\{(\mathcal{I}_{\vartheta,p}f)(z)\}| \geq \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1-\nu)} z^{p-\nu} - \frac{(\vartheta+p)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1-\nu)[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right\} |z|^{p-\nu} \quad (z \in \mathcal{U}; \nu > 0; \vartheta > -p; p, n \in \mathbb{N}) \tag{5.12}$$

and

$$|\mathcal{D}_z^{\nu}\{(\mathcal{I}_{\vartheta,p}f)(z)\}| \leq \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1-\nu)} z^{p-\nu} + \frac{(\vartheta+p)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1-\nu)[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^n \right\} |z|^{p-\nu} \tag{5.13}$$

$$(z \in \mathcal{U}; \nu > 0; \vartheta > -p; p, n \in \mathbb{N}).$$

Each of the assertions (5.12) and (5.13) is sharp.

Proof It follows from Theorem 1 that

$$\sum_{k=n+p}^{\infty} k |a_k| \leq \frac{(n+p)(B-A)(p-\sigma)}{[n(1+B)+(B-A)(p-\sigma)]\Phi_p^{n+p}(\delta, \lambda, \mu, l)}. \tag{5.14}$$

Consider the function $Q(z)$ defined in \mathcal{U} by

$$\begin{aligned} Q(z) &= \frac{\Gamma(p+1-\nu)}{\Gamma(p+1)} z^\nu \mathcal{D}_z^\nu \{(\mathcal{I}_{\vartheta,p} f)(z)\} \\ &= z^p - \sum_{k=n+p}^\infty \frac{(\vartheta+p)\Gamma(k)\Gamma(p+1-\nu)}{(\vartheta+k)\Gamma(k+1-\nu)\Gamma(p+1)} k |a_k| z^k \\ &= z^p - \sum_{k=n+p}^\infty \wp(k) k |a_k| z^k \quad (z \in \mathcal{U}) \end{aligned}$$

where, for convenience,

$$\wp(k) := \frac{(\vartheta+p)\Gamma(k)\Gamma(p+1-\nu)}{(\vartheta+k)\Gamma(k+1-\nu)\Gamma(p+1)} \quad (k \geq p+n; p, n \in \mathbb{N}; 0 \leq \nu < 1). \tag{5.15}$$

Since $\wp(k)$ is a decreasing function of k when $0 \leq \nu < 1$, we find that

$$0 < \wp(k) \leq \wp(n+p) = \frac{(\vartheta+p)\Gamma(n+p)\Gamma(p+1-\nu)}{(\vartheta+n+p)\Gamma(n+p+1-\nu)\Gamma(p+1)} \quad (0 \leq \nu < 1; \vartheta > -p; p, n \in \mathbb{N}). \tag{5.16}$$

Hence, with the aid of (5.14) and (5.16), for all $z \in \mathcal{U}$, we have

$$\begin{aligned} |Q(z)| &\geq |z|^p - \wp(n+p) |z|^{n+p} \sum_{k=n+p}^\infty k |a_k| \geq |z|^p \\ &\quad - \frac{(\vartheta+p)\Gamma(n+p)\Gamma(p+1-\nu)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1-\nu)\Gamma(p+1)[n(1+B)-(B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^{n+p} \end{aligned}$$

and

$$\begin{aligned} |Q(z)| &\leq |z|^p + \wp(n+p) |z|^{n+p} \sum_{k=n+p}^\infty k |a_k| \leq |z|^p \\ &\quad + \frac{(\vartheta+p)\Gamma(n+p)\Gamma(p+1-\nu)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1-\nu)\Gamma(p+1)[n(1+B)-(B-A)(p-\sigma)] \Phi_p^{n+p}(\delta, \lambda, \mu, l)} |z|^{n+p} \end{aligned}$$

which yield the inequalities (5.15) and (5.16) of Theorem 10. Equalities in (5.15) and (5.16) are attained for the function $f(z)$ given by

$$\begin{aligned} &\mathcal{D}_z^\nu \{(\mathcal{I}_{\vartheta,p} f)(z)\} \\ &= \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1-\nu)} - \frac{(\vartheta+p)\Gamma(n+p+1)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1-\nu)(1+B)\Phi_p^{n+p}(\delta, \lambda, \mu, l)} z^n \right\} z^{p-\nu} \end{aligned}$$

or, equivalently, by

$$(\mathcal{I}_{\vartheta,p} f)(z) = z^p - \frac{(\vartheta+p)(B-A)(p-\sigma)}{(\vartheta+n+p)(n+p)(1+B)\Phi_p^{n+p}(\delta, \lambda, \mu, l)} z^{n+p}.$$

Consequently, we complete the proof of Theorem 10. □

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