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Radius Problems for Functions Containing Derivatives of Bessel Functions

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Abstract
In this paper our aim is to find the radii of starlikeness and convexity for three different kinds of normalizations of the function $N_\nu(z) = az^2 J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z)$, where $J_\nu(z)$ is the Bessel function of the first kind of order ν. The key tools in the proof of our main results are the Mittag-Leffler expansion for the function $N_\nu(z)$ and properties of real zeros of it. In addition, by using the Euler-Rayleigh inequalities we obtain some tight lower and upper bounds for the radii of starlikeness and convexity of order zero for the normalized function $N_\nu(z)$. Finally, we evaluate certain multiple sums of the zeros for the function $N_\nu(z)$.

Keywords Normalized Bessel functions of the fist kind · Convex functions · Starlike functions · Zeros of Bessel function derivatives · Radius

Mathematics Subject Classification Primary 33C10; Secondary 30C45

1 Introduction

Denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $r > 0$, the disk of radius r centered at the origin and let $\mathbb{D} = \mathbb{D}_1$. Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D}$ which satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Traditionally, the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. We say that the function $f \in \mathcal{A}$ is starlike in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin. Analytically, the function f is starlike in $\mathbb{D}_r$

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if and only if $\operatorname{Re}\left(zf'(z)/f(z)\right) > 0$, $z \in \mathbb{D}_r$. For $\beta \in [0, 1)$ we say that the function f is starlike of order β in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(zf'(z)/f(z)\right) > \beta$, $z \in \mathbb{D}_r$. We define by the real number

$$r_\beta^*(f) = \sup\left\{r \in \left(0, r_f\right) : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \text{ for all } z \in \mathbb{D}_r\right\}$$

the radius of starlikeness of order β of the function f. Note that $r^*(f) = r_0^*(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r_\beta^*(f)})$ is a starlike domain with respect to the origin.

The function $f \in \mathcal{A}$ is convex in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a convex domain in $\mathbb{C}$. Analytically, the function f is convex in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(1 + zf''(z)/f'(z)\right) > 0$, $z \in \mathbb{D}_r$. For $\beta \in [0, 1)$ we say that the function f is convex of order β in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(1 + zf''(z)f'(z)\right) > \beta$, $z \in \mathbb{D}_r$. The radius of convexity of order β of the function f is defined by the real number

$$r_\beta^c(f) = \sup\left\{r \in \left(0, r_f\right) : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \text{ for all } z \in \mathbb{D}_r\right\}.$$

Note that $r^c(f) = r_0^c(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r_\beta^c(f)})$ is a convex domain.

The Bessel function of the first kind of order ν is defined by [16, p. 217]

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \qquad z \in \mathbb{C}. \tag{1.1}$$

We know that J_ν has all its zeros real for $\nu > -1$. Here now we consider mainly the general function

$$N_\nu(z) = az^2 J_\nu''(z) + bz J_\nu'(z) + c J_\nu(z)$$

studied by Mercer [15]. Here, as in [15], $q = b - a$ and

$$(c = 0 \text{ and } q \neq 0) \text{ or } (c > 0 \text{ and } q > 0).$$

From (1.1), we have the power series representation

$$N_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n Q(2n+\nu)}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \qquad z \in \mathbb{C}. \tag{1.2}$$

where $Q(\nu) = a\nu(\nu-1) + b\nu + c$ $(a, b, c \in \mathbb{R})$. There are three important works concerning the function N_ν. First Mercer's paper [15] in which it is proved that the kth positive zero of N_ν increases with ν in $\nu > 0$. Second, Ismail and Muldoon [10] showed that under the conditions $a, b, c \in \mathbb{R}$ such that $c = 0$ and $b \neq a$ or $c > 0$ and $b > a$;

(i) For $\nu > 0$, the zeros of $N_\nu(z)$ are either real or purely imaginary.
(ii) For $\nu \geq \max\{0, \nu_0\}$, where ν_0 is the largest real root of the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$, the the zeros of $N_\nu(z)$ are real.
(iii) If $\nu > 0$, $Q(\nu)/(b-a) > 0$ and $a/(b-a) < 0$, the zeros of $N_\nu(z)$ are all real except for a single pair which are conjugate purely imaginary.

Lastly, Baricz, Çağlar and Deniz [4] obtained sufficient and necessary conditions for the starlikeness of a normalized form of N_ν by using results of Mercer [15], Ismail and Muldoon [10] and Shah and Trimble [17]. In this paper, we deal with the radii of starlikeness and convexity of order β for the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ defined by (1.3) in the case when $\nu \geq \max\{0, \nu_0\}$. Also we determine tight lower and upper bounds for the radii of starlikeness and convexity of the functions g_ν and h_ν. The key tools in these proofs are some new Mittag-Leffler expansions for quotients of the function N_ν, special properties of the zeros of the function N_ν and their derivatives, Euler-Rayleigh inequalities and the fact that the smallest positive zeros of the functions $g'_\nu(z)$ and $h'_\nu(z)$ are less than the first positive zero of the function N_ν. For recent studies on the geometric properties of Bessel functions, see [1–9, 12, 14, 18].

Note that N_ν does not belong to $\mathcal{A}$. To prove the main results we need normalizations of the function N_ν. In this paper we focus on the following normalized forms

$$f_\nu(z) = \left[\frac{2^\nu \Gamma(\nu+1)}{Q(\nu)} N_\nu(z)\right]^{1/\nu}, \qquad \nu \neq 0, \tag{1.3}$$
$$g_\nu(z) = \frac{2^\nu \Gamma(\nu+1) z^{1-\nu}}{Q(\nu)} N_\nu(z),$$
$$h_\nu(z) = \frac{2^\nu \Gamma(\nu+1) z^{1-\nu/2}}{Q(\nu)} N_\nu(\sqrt{z}).$$

In the rest of this paper, for the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$ we will always assume that $a, b, c \in \mathbb{R}$ ($c = 0$ and $a \neq b$) or ($c > 0$ and $a < b$). Moreover, ν_0 is the largest real root of the quadratic $Q(\nu)$ defined according to the above conditions.

1.1 Zeros of Hyperbolic Polynomials and the Laguerre–Pólya Class of Entire Functions

In this subsection, we recall some necessary information about polynomials and entire functions with real zeros. An algebraic polynomial is called hyperbolic if all its zeros are real. We formulate the following specific statement that we shall need (see [9] for more details).

By definition, a real entire function ψ belongs to the Laguerre–Pó lya class $\mathcal{LP}$ if it can be represented in the form

$$\psi(x) = cx^m e^{-ax^2+\beta x} \prod_{k\geq 1}\left(1+\frac{x}{x_k}\right)e^{-x/x_k},$$

with $c, \beta, x_k \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N} \cup \{0\}$ and $\sum x_k^{-2} < \infty$. Similarly, ϕ is said to be of type $\mathcal{I}$ in the Laguerre-Pólya class, written $\varphi \in \mathcal{LPI}$, if $\phi(x)$ or $\phi(-x)$ can be represented as

$$\phi(x) = cx^m e^{\sigma x} \prod_{k \geq 1} \left(1 + \frac{x}{x_k}\right),$$

with $c \in \mathbb{R}$, $\sigma \geq 0$, $m \in \mathbb{N} \cup \{0\}$, $x_k > 0$ and $\sum x_k^{-1} < \infty$. The class $\mathcal{LP}$ is the complement of the space of hyperbolic polynomials in the topology induced by uniform convergence on compact subsets of the complex plane while $\mathcal{LPI}$ is the complement of the hyperbolic polynomials whose zeros possess a preassigned constant sign. Given an entire function φ with the Maclaurin expansion

$$\varphi(x) = \sum_{k \geq 0} \mu_k \frac{x^k}{k!},$$

its Jensen polynomials are defined by

$$P_m(\varphi; x) = P_m(x) = \sum_{k=0}^{m} \binom{m}{k} \mu_k x^k.$$

The next result of Jensen [11] is a well-known characterization of functions belonging to $\mathcal{LP}$.

Lemma 1.1 *The function φ belongs to $\mathcal{LP}$ ($\mathcal{LPI}$, respectively) if and only if all the polynomials $P_m(\varphi; x)$, $m = 1, 2, \ldots,$ are hyperbolic (hyperbolic with zeros of equal sign). Moreover, the sequence $P_m(\varphi; z/n)$ converges locally uniformly to $\varphi(z)$.*

The following result is a key tool in the proof of our main results.

Lemma 1.2 *If $\nu \geq \max\{0, \nu_0\}$ then the functions*

$$z \mapsto \Psi_\nu(z) = \frac{2^\nu \Gamma(\nu + 1)}{Q(\nu) z^\nu} N_\nu(z)$$

have infinitely many zeros and all of them are positive. Denoting by $\lambda_{\nu,n}$ the nth positive zero of $\Psi_\nu(z)$, under the same conditions the Weierstrassian decomposition

$$\Psi_\nu(z) = \prod_{n \geq 1} \left(1 - \frac{z^2}{\lambda_{\nu,n}^2}\right)$$

is valid, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, if we denote by $\lambda'_{\nu,n}$ the nth positive zero of $\Phi'_\nu(z)$, where $\Phi_\nu(z) =$

$z^\nu \Psi_\nu(z)$, *then the positive zeros of* $\Psi_\nu(z)$ *are interlaced with those of* $\Phi'_\nu(z)$. *In other words, the zeros satisfy the chain of inequalities*

$$\lambda'_{\nu,1} < \lambda_{\nu,1} < \lambda'_{\nu,2} < \lambda_{\nu,2} < \lambda'_{\nu,3} < \lambda_{\nu,3} < \cdots .$$

Proof Recall that in [10] it was proved that if $\nu \geq \max\{0, \nu_0\}$ then $N_\nu(z)$ is an entire function with infinitely many positive zeros. Since the function $N_\nu(z)$ is entire, its infinite product was determined by Baricz et al. [4]. Using the infinite product representation, we get that

$$\frac{\Phi'_\nu(z)}{\Phi_\nu(z)} = \frac{\nu}{z} + \frac{\Psi'_\nu(z)}{\Psi_\nu(z)} = \frac{\nu}{z} + \sum_{n\geq 1} \frac{2z}{z^2 - \lambda^2_{\nu,n}}. \tag{1.4}$$

Differentiating both sides of (1.4), we have

$$\frac{d}{dz}\left(\frac{\Phi'_\nu(z)}{\Phi_\nu(z)}\right) = -\frac{\nu}{z^2} - 2\sum_{n\geq 1} \frac{z^2 + \lambda^2_{\nu,n}}{\left(z^2 - \lambda^2_{\nu,n}\right)^2}, \quad z \neq \lambda_{\nu,n}.$$

The right hand side of the above expression is real and negative for each z real, $\nu \geq \max\{0, \nu_0\}$. Thus, the quotient on the left side of (1.4) is a strictly decreasing function from $+\infty$ to $-\infty$ as z increases through real values over the open interval $\left(\lambda_{\nu,n}, \lambda_{\nu,n+1}\right)$, $n \in \mathbb{N}$. Hence the function $z \mapsto \Phi'_\nu(z)$ vanishes just once between two consecutive zeros of the function $z \mapsto \Phi_\nu(z)$. □

1.2 Euler–Rayleigh Sums for Positive Zeros of $N_\nu(z)$

Baricz et al. [4] proved Mittag–Leffler expansion of $N_\nu(z)$ as follows

$$N_\nu(z) = \alpha z^2 J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z) = \frac{Q(\nu)z^\nu}{2^\nu\Gamma(\nu+1)} \prod_{n\geq 1}\left(1 - \frac{z^2}{\lambda^2_{\nu,n}}\right) \tag{1.5}$$

where $Q(\nu) = a\nu(\nu-1) + b\nu + c$, $a, b, c \in \mathbb{R}$, and $\lambda_{\nu,n}$ is the nth positive zero of $N_\nu(z)$, $n \in \mathbb{N}$. Therefore we can write

$$\begin{aligned} g_\nu(z) &= \frac{2^\nu\Gamma(\nu+1)z^{1-\nu}}{Q(\nu)} N_\nu(z) \\ &= z\prod_{n\geq 1}\left(1 - \frac{z^2}{\lambda^2_{\nu,n}}\right). \end{aligned} \tag{1.6}$$

On the other hand, the series representation of $g_\nu(z)$ is given by

$$g_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n\,\Gamma(\nu+1)Q(2n+\nu)}{n!4^n\Gamma(n+\nu+1)Q(\nu)} z^{2n+1}. \tag{1.7}$$

Now, we would like to mention that by using the Eqs. (1.6) and (1.7) we can obtain the following Euler-Rayleigh sums for the positive zeros of the function g_ν. From the equality (1.7) we have

$$g_\nu(z) = z - \frac{Q(\nu+2)}{4(\nu+1)Q(\nu)} z^3 + \frac{Q(\nu+4)}{32(\nu+1)(\nu+2)Q(\nu)} z^5 - \cdots . \tag{1.8}$$

On the other hand, if we consider (1.6), then some calculations yield that

$$g_\nu(z) = z - \sum_{n\geq 1} \frac{1}{\lambda_{\nu,n}^2} z^3 + \frac{1}{2}\left(\left(\sum_{n\geq 1}\frac{1}{\lambda_{\nu,n}^2}\right)^2 - \sum_{n\geq 1}\frac{1}{\lambda_{\nu,n}^4}\right) z^5 - \cdots . \tag{1.9}$$

By equating the first few coefficients with the same degrees in Eqs. (1.8) and (1.9), we get

$$\sum_{n\geq 1}\frac{1}{\lambda_{\nu,n}^2} = \frac{Q(\nu+2)}{4(\nu+1)Q(\nu)} \tag{1.10}$$

and

$$\sum_{n\geq 1}\frac{1}{\lambda_{\nu,n}^4} = \frac{1}{16(\nu+1)Q(\nu)}\left(\frac{Q^2(\nu+2)}{(\nu+1)Q(\nu)} - \frac{Q(\nu+4)}{(\nu+2)}\right). \tag{1.11}$$

2 Main Results

2.1 Radii of Starlikeness and Convexity of the Functions f_ν, g_ν and h_ν

The first principal result we established concerns the radii of starlikeness and reads as follows.

Theorem 2.1 *Let* $\beta \in [0, 1)$. *The following statements hold:*

(a) If $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$ *then the radius of starlikeness of order* β *of the function* f_ν *is the smallest positive root of the equation*

$$ar^3 J_\nu'''(r) + (2a + b - a\nu\beta)\, r^2 J_\nu''(r) + (b + c - b\nu\beta)\, r J_\nu'(r) - c\nu\beta J_\nu(r) = 0.$$

(b) If $\nu \geq \max\{0, \nu_0\}$, *then the radius of starlikeness of order* β *of the function* g_ν *is the smallest positive root of the equation*

$$(1-\nu) + \frac{ar^3 J_\nu'''(r) + (2a+b)\, r^2 J_\nu''(r) + (b+c)\, r J_\nu'(r)}{ar^2 J_\nu''(r) + br J_\nu'(r) + cJ_\nu(r)} = \beta$$

(c) *If* $\nu \geq \max\{0, \nu_0\}$, *then the radius of starlikeness of order* β *of the function* h_ν *is the smallest positive root of the equation*

$$(2-\nu) + \frac{ar\sqrt{r}J_\nu'''(\sqrt{r}) + (2a+b)\,rJ_\nu''(\sqrt{r}) + (b+c)\sqrt{r}J_\nu'(\sqrt{r})}{arJ_\nu''(\sqrt{r}) + b\sqrt{r}J_\nu'(\sqrt{r}) + cJ_\nu(\sqrt{r})} = 2\beta$$

Proof Firstly, we prove part **a** for $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$ and **b** and **c** for $\nu \geq \max\{0, \nu_0\}$. We need to show that the following inequalities

$$\mathrm{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right) > \beta, \qquad \mathrm{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right) > \beta \quad \text{and} \quad \mathrm{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right) > \beta \tag{2.1}$$

are valid for $z \in \mathbb{D}_{r_\beta^*(f_\nu)}$, $z \in \mathbb{D}_{r_\beta^*(g_\nu)}$ and $z \in \mathbb{D}_{r_\beta^*(h_\nu)}$ respectively, and each of the above inequalities does not hold in any larger disks.

When we write the Eq. (1.5) in the definition of the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ we get by using logarithmic derivation

$$\frac{zf_\nu'(z)}{f_\nu(z)} = \frac{1}{\nu}\frac{zN_\nu'(z)}{N_\nu(z)} = 1 - \frac{1}{\nu}\sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}, \quad \nu \geq \max\{0, \nu_0\},\ \nu \neq 0,$$

$$\frac{zg_\nu'(z)}{g_\nu(z)} = (1-\nu) + \frac{zN_\nu'(z)}{N_\nu(z)} = 1 - \sum_{n\geq 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}, \quad \nu \geq \max\{0, \nu_0\},$$

$$\frac{zh_\nu'(z)}{h_\nu(z)} = \left(1 - \frac{\nu}{2}\right) + \frac{1}{2}\frac{\sqrt{z}N_\nu'(\sqrt{z})}{N_\nu(\sqrt{z})} = 1 - \sum_{n\geq 1}\frac{z}{\lambda_{\nu,n}^2 - z}, \quad \nu \geq \max\{0, \nu_0\}.$$

It is known [2] that if $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are such that $\lambda > |z|$, then

$$\frac{|z|}{\lambda - |z|} \geq \mathrm{Re}\left(\frac{z}{\lambda - z}\right). \tag{2.2}$$

Then the inequality

$$\frac{|z|^2}{\lambda_{\nu,n}^2 - |z|^2} \geq \mathrm{Re}\left(\frac{z^2}{\lambda_{\nu,n}^2 - z^2}\right)$$

holds for every $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$ and $|z| < \lambda_{\nu,1}$. Therefore,

$$\mathrm{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right) = 1 - \frac{1}{\nu}\sum_{n\geq 1}\mathrm{Re}\left(\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right) \geq 1 - \frac{1}{\nu}\sum_{n\geq 1}\frac{2|z|^2}{\lambda_{\nu,n}^2 - |z|^2} = \frac{|z|\,f_\nu'(|z|)}{f_\nu(|z|)},$$

$$\mathrm{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right) = 1 - \sum_{n\geq 1}\mathrm{Re}\left(\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right) \geq 1 - \sum_{n\geq 1}\frac{2|z|^2}{\lambda_{\nu,n}^2 - |z|^2} = \frac{|z|\,g_\nu'(|z|)}{g_\nu(|z|)},$$

$$\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right) = 1-\sum_{n\geq 1}\operatorname{Re}\left(\frac{z}{\lambda_{\nu,n}^2-z}\right) \geq 1-\sum_{n\geq 1}\frac{|z|}{\lambda_{\nu,n}^2-|z|} = \frac{|z|\,h_\nu'(|z|)}{h_\nu(|z|)},$$

where equalities are attained only when $z=|z|=r$. Thus, for $r\in\left(0,\lambda_{\nu,1}\right)$ it follows that

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)\right\} = \frac{rf_\nu'(r)}{f_\nu(r)},$$
$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right)\right\} = \frac{rg_\nu'(r)}{g_\nu(r)},$$
$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)\right\} = \frac{rh_\nu'(r)}{h_\nu(r)}.$$

On the other and, the mappings $\psi_\nu,\ \varphi_\nu,\ \phi_\nu:\left(0,\lambda_{\nu,1}\right)\to\mathbb{R}$ defined by

$$\psi_\nu(r) = \frac{rf_\nu'(r)}{f_\nu(r)} = 1-\frac{1}{\nu}\sum_{n\geq 1}\left(\frac{2r^2}{\lambda_{\nu,n}^2-r^2}\right),$$
$$\varphi_\nu(r) = \frac{rg_\nu'(r)}{g_\nu(r)} = 1-\sum_{n\geq 1}\left(\frac{2r^2}{\lambda_{\nu,n}^2-r^2}\right),$$
$$\phi_\nu(r) = \frac{rh_\nu'(r)}{h_\nu(r)} = 1-\sum_{n\geq 1}\left(\frac{r}{\lambda_{\nu,n}^2-r}\right)$$

are strictly decreasing since

$$\psi_\nu'(r) = -\frac{1}{\nu}\sum_{n\geq 1}\left(\frac{4r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2-r^2\right)^2}\right) < 0,$$
$$\varphi_\nu'(r) = -\sum_{n\geq 1}\left(\frac{4r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2-r^2\right)^2}\right) < 0,$$
$$\phi_\nu'(r) = -\sum_{n\geq 1}\left(\frac{\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2-r\right)^2}\right) < 0$$

for all $\nu\geq\max\{0,\nu_0\},\ \nu\neq 0$. Now, since

$$\lim_{r\searrow 0}\psi_\nu(r)=1>\beta,\quad \lim_{r\searrow 0}\varphi_\nu(r)=1>\beta,\quad \lim_{r\searrow 0}\phi_\nu(r)=1>\beta,$$
$$\lim_{r\nearrow\lambda_{\nu,1}}\psi_\nu(r)=-\infty,\quad \lim_{r\nearrow\lambda_{\nu,1}}\varphi_\nu(r)=-\infty,\quad \lim_{r\nearrow\lambda_{\nu,1}}\phi_\nu(r)=-\infty,$$

in view of the minimum principle for harmonic functions imply that the corresponding inequalities in (2.1) for $\nu\geq\max\{0,\nu_0\}$, $\nu\neq 0$ hold if and only if $z\in\mathbb{D}_{r_1}$, $z\in\mathbb{D}_{r_2}$

Table 1 Radii of starlikeness for f_ν when $\nu = 1.5$

	$r^*_\beta(f_{1.5})$							
	$b = 1$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$	
	$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$
$a = 2$	0.8231	0.6458	$b = 2$	1.0917	0.8481	$c = 2$	1.3089	1.0058
$a = 3$	0.7689	0.6045	$b = 3$	1.1774	0.9120	$c = 3$	1.3952	1.0671
$a = 4$	0.7382	0.5809	$b = 4$	1.2337	0.9539	$c = 4$	1.4708	1.1203

Table 2 Radii of starlikeness for g_ν when $\nu = 1.5$

	$r^*_\beta(g_{1.5})$							
	$b = 1$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$	
	$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$
$a = 2$	0.7188	0.5483	$b = 2$	0.9477	0.7167	$c = 2$	1.1285	0.8459
$a = 3$	0.6723	0.5137	$b = 3$	1.0203	0.7697	$c = 3$	1.1995	0.8957
$a = 4$	0.6458	0.4939	$b = 4$	1.0678	0.8044	$c = 4$	1.2611	0.9388

Table 3 Radii of starlikeness for h_ν when $\nu = 1.5$

	$r^*_\beta(h_{1.5})$							
	$b = 1$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$	
	$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$
$a = 2$	0.8009	0.5167	$b = 2$	1.4211	0.8982	$c = 2$	2.0638	1.2737
$a = 3$	0.6979	0.4520	$b = 3$	1.6575	1.0410	$c = 3$	2.3560	1.4388
$a = 4$	0.6426	0.4171	$b = 4$	1.8225	1.1403	$c = 4$	2.6296	1.5905

and $z \in \mathbb{D}_{r_3}$, respectively, where r_1, r_2 and r_3 are the smallest positive roots of the equations

$$\frac{rf'_\nu(r)}{f_\nu(r)} = \beta, \qquad \frac{rg'_\nu(r)}{g_\nu(r)} = \beta, \qquad \frac{rh'_\nu(r)}{h_\nu(r)} = \beta$$

which are equivalent to

$$\frac{rN'_\nu(r)}{\nu N_\nu(r)} = \beta, \qquad (1-\nu) + \frac{rN'_\nu(r)}{N_\nu(r)} = \beta, \quad \left(1 - \frac{\nu}{2}\right) + \frac{1}{2}\frac{\sqrt{r}N'_\nu(\sqrt{r})}{N_\nu(\sqrt{r})} = \beta$$

situated in $\left(0, \lambda_{\nu,1}\right)$.

This completes the proof of part **a** when $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$, and parts **b** and **c** when $\nu \geq \max\{0, \nu_0\}$. □

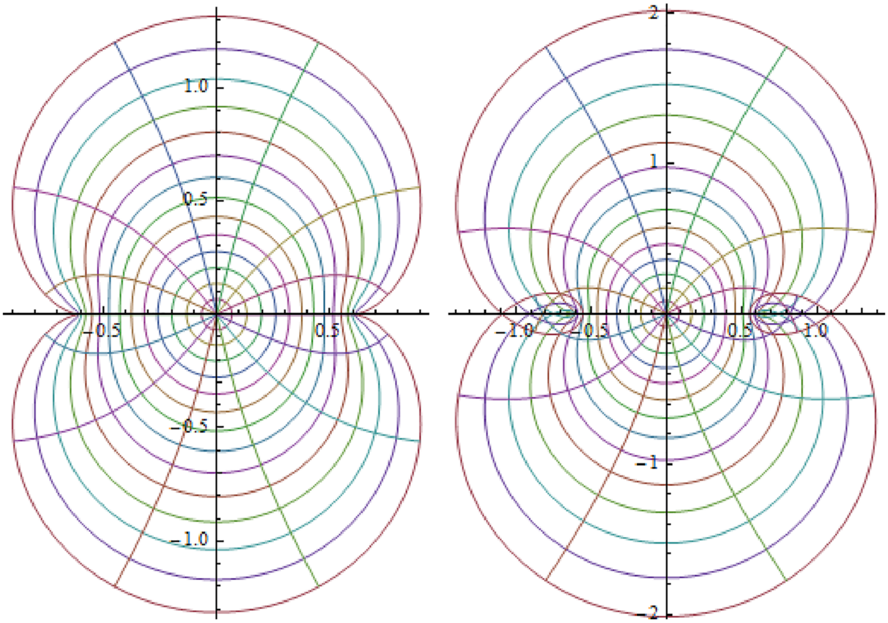

Fig. 1 Images of function $g_{1.5}(z)$ in $\mathbb{D}_{0.9477}$ and $\mathbb{D}_{1.2}$, respectively

When $\nu = 1.5$, considering the special values of $a, b, c \in \mathbb{R}$, radii of starlikeness of the functions f_ν, g_ν and h_ν is seen from Tables 1, 2 and 3. If the values of b and c are fixed and the value of a is increased, then radii of starlikeness of the functions f_ν, g_ν and h_ν are monotone decreasing. If the values of a and c are fixed and the value of b is increased or the values of a and b are fixed and the value of c is increased then radii of starlikeness of the functions f_ν, g_ν and h_ν are monotone increasing. In addition, according to the increasing values of β, it is clear that radii of starlikeness of the functions f_ν, g_ν and h_ν are monotone decreasing (Fig. 1).

The second principal result we established concerns the radii of convexity and reads as follows.

Theorem 2.2 *Let $\beta \in [0, 1)$. The following statements hold:*

(a) If $\nu \geq \max\{0, \nu_0\}$, $\nu \neq 0$ then, the radius $r^c_\beta(f_\nu)$ is the smallest positive root of the equation

$$1 + \frac{rN''_\nu(r)}{N'_\nu(r)} + \left(\frac{1}{\nu} - 1\right)\frac{rN'_\nu(r)}{N_\nu(r)} = \beta$$

Moreover, $r^c_\beta(f_\nu) < \lambda'_{\nu,1} < \lambda_{\nu,1}$, where $\lambda_{\nu,1}$ and $\lambda'_{\nu,1}$ denote the first positive zeros of N_ν and N'_ν, respectively.

(b) *If* $\nu \geq \max\{0, \nu_0\}$ *then, the radius* $r_\beta^c(g_\nu)$ *is the smallest positive root of the equation*

$$1+\frac{r^2 N_\nu''(r)+(2-2\nu)\, r N_\nu'(r)+\left(\nu^2-\nu\right) N_\nu(r)}{r N_\nu'(r)+(1-\nu)\, N_\nu(r)}=\beta .$$

(c) *If* $\nu \geq \max\{0, \nu_0\}$ *then, the radius* $r_\beta^c(h_\nu)$ *is the smallest positive root of the equation*

$$1+\frac{r N_\nu''(\sqrt{r})+(3-2\nu)\sqrt{r} N_\nu'(\sqrt{r})+\left(\nu^2-2\nu\right) N_\nu(\sqrt{r})}{2\sqrt{r} N_\nu'(\sqrt{r})+2\,(2-\nu)\, N_\nu(\sqrt{r})}=\beta .$$

Proof **a)** Since

$$1+\frac{z f_\nu''(z)}{f_\nu'(z)}=1+\frac{z N_\nu''(z)}{N_\nu'(z)}+\left(\frac{1}{\nu}-1\right) \frac{z N_\nu'(z)}{N_\nu(z)}$$

and by means of (1.5) we have

$$\frac{z N_\nu'(z)}{N_\nu(z)}=\nu-\sum_{n \geq 1} \frac{2 z^2}{\lambda_{\nu, n}^2-z^2} .$$

Moreover, we obtain

$$N_\nu'(z)=\sum_{n \geq 0} \frac{(-1)^n\,(2 n+\nu)\, Q(2 n+\nu)}{2 . n!\, \Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2 n+\nu}, \qquad z \in \mathbb{C}$$

and

$$\frac{2^\nu \Gamma\,(\nu+1)}{Q(\nu) z^{\nu-1} \nu} N_\nu'(z)=1+\sum_{n \geq 0} \frac{(-1)^n\,(2 n+\nu)\, Q(2 n+\nu)}{2 . n!\, \Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2 n+\nu}, \qquad z \in \mathbb{C} .$$

Taking into consideration the well-known limit

$$\lim _{n \rightarrow \infty} \frac{\log \Gamma\,(n+c)}{n \log n}=1,$$

where c is a positive constant, and Levin [13] we infer that above entire function is of growth order $\rho=1/2$. Namely, for $\nu \geq \max\{0, \nu_0\},\ \nu \neq 0$ we have

$$\lim _{n \rightarrow \infty} \frac{n \log n}{n \log 4+\log \Gamma(n+\nu+1)+\log \Gamma(n+1)-\log Q(2 n+\nu-\log (2 n+\nu)}=\frac{1}{2} .$$

Now, by applying Hadamard's Theorem [13] we can write the infinite product representation of $N'_\nu(z)$ as follows:

$$\frac{Q(\nu)z^{\nu-1}\nu}{2^\nu\Gamma(\nu+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\lambda'^2_{\nu,n}}\right)$$

where $\lambda'_{\nu,n}$ denotes the nth positive zero of the function $N'_\nu(z)$.

Observe also that

$$1+\frac{zf''_\nu(z)}{f'_\nu(z)}=1-\left(\frac{1}{\nu}-1\right)\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\nu,n}-z^2}-\sum_{n\geq 1}\frac{2z^2}{\lambda'^2_{\nu,n}-z^2}.$$

Now, by using the inequality (2.2), for all $z\in\mathbb{D}_{\lambda'_{\nu,1}}$ and $1\geq\nu\geq\max\{0,\nu_0\},\ \nu\neq 0$ we obtain the inequality

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zf''_\nu(z)}{f'_\nu(z)}\right)&=1-\left(\frac{1}{\nu}-1\right)\sum_{n\geq 1}\operatorname{Re}\left(\frac{2z^2}{\lambda^2_{\nu,n}-z^2}\right)-\sum_{n\geq 1}\operatorname{Re}\left(\frac{2z^2}{\lambda'^2_{\nu,n}-z^2}\right)\\&\geq 1-\left(\frac{1}{\nu}-1\right)\sum_{n\geq 1}\frac{2r^2}{\lambda^2_{\nu,n}-r^2}-\sum_{n\geq 1}\frac{2r^2}{\lambda'^2_{\nu,n}-r^2},\end{aligned}$$

where $|z|=r$. Moreover, observe that if we use the inequality [2, Lemma 2.1]

$$\mu\operatorname{Re}\left(\frac{z}{a-z}\right)-\operatorname{Re}\left(\frac{z}{b-z}\right)\geq\mu\frac{|z|}{a-|z|}-\frac{|z|}{b-|z|}$$

where $a>b>0$, $\mu\in[0,1]$ and $z\in\mathbb{C}$ such that $|z|<b$, then we get that the above inequality is also valid when $\nu\geq 1$. Here we used that the zeros of N_ν and N'_ν are interlacing according to Lemma 1.2. The above inequality implies for $r\in(0,\lambda'_{\nu,1})$

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zf''_\nu(z)}{f'_\nu(z)}\right)\right\}=1+\frac{rf''_\nu(r)}{f'_\nu(r)}.$$

On the other hand, we define the function $\Lambda_\nu:(0,\lambda'_{\nu,1})\to\mathbb{R}$,

$$\Lambda_\nu(r)=1+\frac{rf''_\nu(r)}{f'_\nu(r)}.$$

Since the zeros of N_ν and N'_ν are interlacing according to Lemma 1.2 and $r<\lambda'_{\nu,1}<\lambda_{\nu,1}$ (or $r<\sqrt{\lambda_{\nu,1}\lambda'_{\nu,1}}$) for all $\nu\geq\max\{0,\nu_0\},\ \nu\neq 0$ we have

$$\left(\lambda_{\nu,n}\right)\left(\lambda'^2_{\nu,n}-r^2\right)-\left(\lambda'_{\nu,n}\right)\left(\lambda^2_{\nu,n}-r^2\right)<0.$$

Thus the following inequality

$$\begin{aligned}\Lambda_\nu'(r) &= -\left(\frac{1}{\nu}-1\right)\sum_{n\geq 1}\frac{4r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq 1}\frac{4r\lambda_{\nu,n}'^2}{\left(\lambda_{\nu,n}'^2-r^2\right)^2}\\ &< \sum_{n\geq 1}\frac{4r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq 1}\frac{4r\lambda_{\nu,n}'^2}{\left(\lambda_{\nu,n}'^2-r^2\right)^2}\\ &= 4r\sum_{n\geq 1}\frac{\left(\lambda_{\nu,n}\right)^2\left(\lambda_{\nu,n}'^2-r^2\right)^2-\left(\lambda_{\nu,n}'\right)^2\left(\lambda_{\nu,n}^2-r^2\right)^2}{\left(\lambda_{\nu,n}^2-r^2\right)^2\left(\lambda_{\nu,n}'^2-r^2\right)^2}<0\end{aligned}$$

is satisfied. Consequently, the function Λ_ν is strictly decreasing. Observe also that $\lim_{r\searrow 0}\Lambda_\nu(r)=1>\beta$ and $\lim_{r\nearrow\lambda_{\nu,1}'}\Lambda_\nu(r)=-\infty$, which means that for $z\in\mathbb{D}_{r_4}$ we have

$$\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)>\beta$$

if and only if r_4 is the unique root of

$$1+\frac{rf_\nu''(r)}{f_\nu'(r)}=\beta \text{ or } 1+\frac{rN_\nu''(r)}{N_\nu'(r)}+\left(\frac{1}{\nu}-1\right)\frac{rN_\nu'(r)}{N_\nu(r)}=\beta,$$

situated in $(0,\lambda_{\nu,1}')$.

b) Observe that

$$1+\frac{zg_\nu''(z)}{g_\nu'(z)}=1+\frac{z^2N_\nu''(z)+2(1-\nu)zN_\nu'(z)+(\nu^2-\nu)N_\nu(z)}{zN_\nu'(z)+(1-\nu)N_\nu(z)}.$$

By using (1.3) and (1.5) we have

$$\begin{aligned}g_\nu'(z) &= \frac{2^\nu\Gamma(\nu+1)z^{-\nu}}{Q(\nu)}\left[(1-\nu)N_\nu(z)+zN_\nu'(z)\right]\\ &= \sum_{n\geq 0}\frac{(-1)^n\,(2n+1)Q(2n+\nu)\Gamma(\nu+1)}{n!\Gamma(n+\nu+1)Q(\nu)}\left(\frac{z}{2}\right)^{2n}\end{aligned}\tag{2.3}$$

and

$$\lim_{n\to\infty}\frac{n\log n}{n\log 4+\log\Gamma(n+\nu+1)+\log\Gamma(n+1)+\log Q(\nu)-\log Q(2n+\nu)-\log(2n+1)-\log\Gamma(\nu+1)}$$
$$=\frac{1}{2}.$$

Here, we used

$$n!=\Gamma(n+1)\qquad\text{and}\qquad\lim_{n\to\infty}\frac{\log\Gamma(an+b)}{n\log n}=a,$$

where b and c are positive constants. So, by applying Hadamard's Theorem [13, p. 26] we can write the infinite product representation of $g_\nu'(z)$ as follows:

$$g_\nu'(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\delta_{\nu,n}^2}\right), \tag{2.4}$$

where $\delta_{\nu,n}$ denotes the nth positive zero of the function g_ν'. From Lemma 1.2 for $\nu > \max\{0, \nu_0\}$ the function $g_\nu' \in \mathcal{LP}$, and the smallest positive zero of g_ν' does not exceed the first positive zero of N_ν.

By means of (2.4) we have

$$1+\frac{zg_\nu''(z)}{g_\nu'(z)} = 1-\sum_{n\geq 1}\left(\frac{2z^2}{\delta_{\nu,n}^2-z^2}\right).$$

By using the inequality (2.2), for all $z \in \mathbb{D}_{\delta_{\nu,n}}$ we obtain the inequality

$$\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right) \geq 1-\sum_{n\geq 1}\frac{2r^2}{\delta_{\nu,n}^2-r^2}$$

where $|z| = r$. Thus, for $r \in (0, \delta_{\nu,1})$ we get

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)\right\} = 1+\frac{rg_\nu''(r)}{g_\nu'(r)}.$$

The function $\Theta_\nu : (0, \delta_{\nu,1}) \to \mathbb{R}$, defined by

$$\Theta_\nu(r) = 1+\frac{rg_\nu''(r)}{g_\nu'(r)},$$

is strictly decreasing and $\lim_{r\searrow 0}\Theta_\nu(r) = 1 > \beta$ and $\lim_{r\nearrow\delta_{\nu,1}}\Theta_{\nu,n}(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_5}$ we have that

$$\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right) > \beta$$

if and ony if r_5 is the unique root of

$$1+\frac{rg_\nu''(r)}{g_\nu'(r)} = \beta,$$

situated in $(0, \delta_{\nu,1})$.

c) Observe that

$$1+\frac{zh_\nu''(z)}{h_\nu'(z)}=1+\frac{zN_\nu''(\sqrt{z})+(3-2\nu)\sqrt{z}N_\nu'(\sqrt{z})+\left(\nu^2-2\nu\right)N_\nu(\sqrt{z})}{2\sqrt{z}N_\nu'(\sqrt{z})+2\,(2-\nu)\,N_\nu(\sqrt{z})}.$$

By using (1.3) and (1.5) we have that

$$\begin{aligned} h_\nu'(z)&=\frac{2^{\nu-1}\Gamma(\nu+1)z^{-\nu/2}}{Q\,(\nu)}\left[(2-\nu)N_\nu(\sqrt{z})+\sqrt{z}N_\nu'(\sqrt{z})\right]\\ &=\sum_{n\geq 0}\frac{(-1)^n\,(n+1)\Gamma(\nu+1)Q\,(2n+\nu)}{n!\Gamma(n+\nu+1)Q\,(\nu)}\left(\frac{z}{4}\right)^n \end{aligned} \tag{2.5}$$

and

$$\lim_{n\to\infty}\frac{n\log n}{n\log 4+\log\Gamma(n+\nu+1)+\log\Gamma(n+1)+\log Q(\nu)-\log Q(2n+\nu)-\log(n+1)-\log\Gamma(\nu+1)}$$
$$=\frac{1}{2}.$$

So, by applying Hadamard's Theorem [13, p. 26] we can write the infinite product representation of $h_\nu'(z)$ as follows:

$$h_\nu'(z)=\prod_{n\geq 1}\left(1-\frac{z}{\gamma_{\nu,n}^2}\right), \tag{2.6}$$

where $\gamma_{\nu,n}$ denotes the nth positive zero of the function $h_{\nu,n}'$. From Lemma 1.2 for $\nu\geq\max\{0,\nu_0\}$ the function $h_\nu'\in\mathcal{LP}$, and the smallest positive zero of h_ν' does not exceed the first positive zero of N_ν.

By means of (2.4) we have

$$1+\frac{zh_\nu''(z)}{h_\nu'(z)}=1-\sum_{n\geq 1}\frac{z}{\gamma_{\nu,n}^2-z}.$$

By using the inequality (2.2), for all $z\in\mathbb{D}_{\gamma_{\nu,n}}$ we obtain the inequality

$$\mathrm{Re}\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)\geq 1-\sum_{n\geq 1}\frac{r}{\gamma_{\nu,n}^2-r},$$

where $|z|=r$. Thus, for $r\in(0,\gamma_{\nu,1})$ we get

$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)\right\}=1+\frac{rh_\nu''(r)}{h_\nu'(r)}.$$

Table 4 Radii of convexity for f_ν when $\nu = 2.5$

	$r^c_\beta(f_{2.5})$							
	$b = 1$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$	
	$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$
$a = 2$	1.0057	0.7896	$b = 2$	1.1515	0.8992	$c = 2$	1.2412	0.9653
$a = 3$	0.9810	0.7709	$b = 3$	1.2069	0.9408	$c = 3$	1.2800	0.9938
$a = 4$	0.9676	0.7607	$b = 4$	1.2460	0.9702	$c = 4$	1.3155	1.0197

Table 5 Radii of convexity for g_ν when $\nu = 2.5$

	$r^c_\beta(g_{2.5})$							
	$b = 1$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$	
	$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$
$a = 2$	0.6839	0.5219	$b = 2$	0.7769	0.5913	$c = 2$	0.8325	0.6323
$a = 3$	0.6680	0.5101	$b = 3$	0.8122	0.6176	$c = 3$	0.8563	0.6498
$a = 4$	0.6594	0.5036	$b = 4$	0.8371	0.6361	$c = 4$	0.8780	0.6658

The function $\Upsilon_\nu : (0, \gamma_{\nu,1}) \to \mathbb{R}$, defined by

$$\Upsilon_\nu(r) = 1 + \frac{rh''_\nu(r)}{h'_\nu(r)},$$

is strictly decreasing and $\lim_{r\searrow 0} \Upsilon_\nu(r) = 1 > \beta$ and $\lim_{r\nearrow \gamma_{\nu,1}} \Upsilon_\nu(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_6}$ we have that

$$\operatorname{Re}\left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)}\right) > \beta$$

if and ony if r_6 is the unique root of

$$1 + \frac{rh''_\nu(r)}{h'_\nu(r)} = \beta,$$

situated in $(0, \gamma_{\nu,1})$. □

In Tables 4, 5, and 6, the convexity radii according to special values of $a, b, c \in \mathbb{R}$ showed a monotony similar to the radii of starlikeness.

Table 6 Radii of convexity for h_ν when $\nu = 2.5$

	$r^c_\beta(h_{2.5})$							
	$b = 1$ and $c = 0$			$a = 1$ and $c = 0$			$a = 1$ and $b = 2$	
	$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$		$\beta = 0$	$\beta = 0.5$
$a = 2$	1.4835	1.0997	$b = 2$	1.9725	1.4489	$c = 2$	2.3191	1.6906
$a = 3$	1.4080	1.0453	$b = 3$	2.1779	1.5946	$c = 3$	2.4797	1.8014
$a = 4$	1.3681	1.0165	$b = 4$	2.3289	1.7016	$c = 4$	2.6322	1.9060

2.2 Bounds for Radii of Starlikeness and Convexity of The Functions g_ν and h_ν

In this subsection we consider two different functions g_ν and h_ν which are normalized forms of the function N_ν given by (1.3). Here firstly our aim is to show that the radii of univalence of these functions correspond to the radii of starlikeness.

Theorem 2.3 *The following statements hold:*

(a) If $\nu \geq \max\{0, \nu_0\}$, *then* $r^*(g_\nu)$ *satisfies the inequalities*

$$r^*(g_\nu) < \sqrt{\frac{2(\nu+1)Q(\nu)}{Q(\nu+2)}},$$

$$2\sqrt{\frac{(\nu+1)Q(\nu)}{3Q(\nu+2)}} < r^*(g_\nu) < 2\sqrt{\frac{Q(\nu+2)}{\frac{3Q^2(\nu+2)}{(\nu+1)Q(\nu)} - \frac{5Q(\nu+2)}{3(\nu+2)}}}$$

and

$$\sqrt[4]{\frac{16(\nu+1)^2(\nu+2)Q^2(\nu)}{9(\nu+2)Q^2(\nu+2) - 5(\nu+1)Q(\nu)Q(\nu+4)}} < r^*(g_\nu)$$
$$< \sqrt{\frac{8(\nu+1)(\nu+3)Q(\nu)\left[9(\nu+2)Q^2(\nu+2) - 5(\nu+1)Q(\nu)Q(\nu+4)\right]}{9(\nu+3)Q(\nu+2)\left[6(\nu+2)Q^2(\nu+2) - 5(\nu+1)^2Q(\nu)Q(\nu+4)\right] + 7(\nu+1)^2Q^2(\nu)Q(\nu+6)}}.$$

(b) If $\nu \geq \max\{0, \nu_0\}$, *then* $r^*(h_\nu)$ *satisfies the inequalities*

$$r^*(h_\nu) < \frac{2(\nu+1)Q(\nu)}{Q(\nu+2)},$$

$$\frac{2(\nu+1)Q(\nu)}{3Q(\nu+2)} < r^*(h_\nu) < \frac{2Q(\nu+2)}{\frac{Q^2(\nu+2)}{(\nu+1)Q(\nu)} - \frac{3Q(\nu+4)}{4(\nu+2)}}$$

and

$$\sqrt{\frac{4(\nu+1)Q(\nu)}{\frac{Q^2(\nu+2)}{(\nu+1)Q(\nu)} - \frac{3Q(\nu+4)}{4(\nu+2)}}} < r^*(h_\nu)$$

$$< \frac{4(\nu+3)Q(\nu)\left[-4(\nu+2)Q^2(\nu+2)+3(\nu+1)^2Q(\nu)Q(\nu+4)\right]}{(\nu+3)Q(\nu+2)\left[8(\nu+2)Q^2(\nu+2)-9(\nu+1)Q(\nu)Q(\nu+4)\right]+2(\nu+1)^2Q^2(\nu)Q(\nu+6)}$$

Proof **a)** By using the first Rayleigh sum (1.10) and the implict relation for $r^*(g_\nu)$, obtained by Kreyszing and Todd [12], we get that

$$\frac{1}{(r^*(g_\nu))^2} = \sum_{n\geq 1} \frac{2}{\lambda_{\nu,n}^2 - (r^*(g_\nu))^2} > \sum_{n\geq 1} \frac{2}{\lambda_{\nu,n}^2} = \frac{Q(\nu+2)}{2(\nu+1)Q(\nu)}.$$

Now, by using the Euler-Rayleigh inequalities it is possible to get tighter bounds for the radius of univalence (and starlikeness) $r^*(g_\nu)$. We define the function $G_\nu(z) = (z\Psi_\nu(z))' = g_\nu'(z)$, where g_ν' is defined by (2.4). Now, taking the logarithmic derivative of both sides of (2.4) for $|z| < \delta_{\nu,1}$ we have

$$\begin{aligned} \frac{G_\nu'(z)}{G_\nu(z)} &= -\sum_{n\geq 1} \frac{2z}{\delta_{\nu,n}^2 - z^2} = -2\sum_{n\geq 1}\sum_{k\geq 0} \frac{1}{\left(\delta_{\nu,n}\right)^{2(k+1)}} z^{2k+1} \\ &= -2\sum_{k\geq 0} \sigma_{k+1} z^{2k+1} \end{aligned} \tag{2.7}$$

where $\sigma_k = \sum_{n\geq 1} \left(\delta_{\nu,n}\right)^{-2k}$ is the Euler-Rayleigh sum for the zeros of G_ν. Also, using (2.3) from the infinite sum representation of G_ν we obtain

$$\frac{G_\nu'(z)}{G_\nu(z)} = \frac{\sum_{n\geq 0} U_n z^{2n+1}}{\sum_{n\geq 0} V_n z^{2n}}, \tag{2.8}$$

where

$$U_n = \frac{2\,(-1)^{n+1}\,\Gamma(\nu+1)Q(2n+\nu+2)(2n+3)}{n!4^{n+1}\Gamma(n+\nu+2)Q(\nu)}$$

and

$$V_n = \frac{(-1)^n\,\Gamma(\nu+1)Q(2n+\nu)(2n+1)}{n!4^n\Gamma(n+\nu+1)Q(\nu)}.$$

By comparing the coefficients with the same degrees as (2.7) and (2.8) we obtain the Euler-Rayleigh sums

$$\sigma_1 = \frac{3Q(\nu+2)}{4(\nu+1)Q(\nu)} \text{ and } \sigma_2 = \frac{9(\nu+2)Q^2(\nu+2)-5(\nu+1)Q(\nu)Q(\nu+4)}{16(\nu+1)^2(\nu+2)Q^2(\nu)}$$

and

$$\sigma_3 = \frac{54(\nu+2)(\nu+3)Q^3(\nu+2)-45(\nu+1)^2(\nu+3)Q(\nu)Q(\nu+2)Q(\nu+4)+7(\nu+1)^2Q^2(\nu)Q(\nu+6)}{128(\nu+1)^3(\nu+2)(\nu+3)Q^3(\nu)}$$

By using the Euler-Rayleigh inequalities

$$\sigma_k^{-1/k} < \delta_{\nu,1}^2 < \frac{\sigma_k}{\sigma_{k+1}}$$

for $\nu \geq \max\{0, \nu_0\}$, $k \in \mathbb{N}$ and for $k = 1$ and $k = 2$ we get the following inequalities

$$\frac{4(\nu+1)Q(\nu)}{3Q(\nu+2)} < \left(r^*(g_\nu)\right)^2 < \frac{4Q(\nu+2)}{\frac{3Q^2(\nu+2)}{(\nu+1)Q(\nu)} - \frac{5Q(\nu+4)}{3(\nu+2)}}$$

and

$$\sqrt{\frac{16(\nu+1)^2(\nu+2)Q^2(\nu)}{9(\nu+2)Q^2(\nu+2) - 5(\nu+1)Q(\nu)Q(\nu+4)}} < \left(r^*(g_\nu)\right)^2 <$$
$$< \frac{8(\nu+1)(\nu+3)Q(\nu)\left[9(\nu+2)Q^2(\nu+2) - 5(\nu+1)Q(\nu)Q(\nu+4)\right]}{9(\nu+3)Q(\nu+2)\left[6(\nu+2)Q^2(\nu+2) - 5(\nu+1)^2Q(\nu)Q(\nu+4)\right] + 7(\nu+1)^2Q^2(\nu)Q(\nu+6)}$$

and it is possible to get tighter bounds for other values of $k \in \mathbb{N}$.

b) By using the first Rayleigh sum (1.10) and the implict relation for $r^*(h_\nu)$, obtained by Kreyszing and Todd [12], we get for all $\nu \geq \max\{0, \nu_0\}$ that

$$\frac{1}{r^*(h_\nu)} = \sum_{n\geq 1} \frac{1}{\lambda_{\nu,n}^2 - r^*(h_\nu)} > \sum_{n\geq 1} \frac{1}{\lambda_{\nu,n}^2} = \frac{Q(\nu+2)}{2(\nu+1)Q(\nu)}.$$

Now, by using the Euler-Rayleigh inequalities it is possible to get tighter bounds for the radius of univalence (and starlikeness) $r^*(h_\nu)$. We define the function $H_\nu(z) = \left(z\Psi_\nu(\sqrt{z})\right)' = h_\nu'(z)$, where h_ν' is defined by (2.5) or (2.6). Now, taking the logarithmic derivative of both sides of (2.6) we have

$$\begin{aligned}\frac{H_\nu'(z)}{H_\nu(z)} &= -\sum_{n\geq 1} \frac{1}{\gamma_{\nu,n} - z} = -\sum_{n\geq 1}\sum_{k\geq 0} \frac{1}{\left(\gamma_{\nu,n}\right)^{k+1}} z^k \\ Q &= -\sum_{k\geq 0} \rho_{k+1} z^k, \qquad |z| < \gamma_{\nu,1},\end{aligned} \tag{2.9}$$

where $\rho_k = \sum_{n\geq 1}\left(\gamma_{\nu,n}\right)^{-k}$ is the Euler-Rayleigh sum for the zeros of H_ν. Also, using (2.5) from the infinite sum representation of H_ν we obtain

$$\frac{H_\nu'(z)}{H_\nu(z)} = \frac{\sum\limits_{n\geq 0} K_n z^n}{\sum\limits_{n\geq 0} L_n z^n}, \tag{2.10}$$

where

$$K_n = \frac{(-1)^{n+1}\Gamma(\nu+1)Q(2n+\nu+2)(n+2)}{n!4^{n+1}\Gamma(n+\nu+2)Q(\nu)}$$

and

$$L_n = \frac{(-1)^n \Gamma(\nu+1) Q(2n+\nu)(n+1)}{n! 4^n \Gamma(n+\nu+1) Q(\nu)}.$$

By comparing the coefficients with the same degrees as (2.9) and (2.10) we obtain the Euler-Rayleigh sums

$$\rho_1 = \frac{Q(\nu+2)}{2(\nu+1)Q(\nu)},$$
$$\rho_2 = \frac{1}{4(\nu+1)Q(\nu)} \left(\frac{Q^2(\nu+2)}{(\nu+1)Q(\nu)} - \frac{3Q(\nu+4)}{4(\nu+2)} \right),$$
$$\rho_3 = \frac{1}{64(\nu+1)^3 Q^3(\nu)} \left(8Q^3(\nu+2) - \frac{9(\nu+1)Q(\nu)Q(\nu+2)Q(\nu+4)}{(\nu+2)} + \frac{2(\nu+1)^2 Q^2(\nu)Q(\nu+6)}{(\nu+2)(\nu+3)} \right)$$

By using the Euler-Rayleigh inequalities

$$\rho_k^{-1/k} < \gamma_{\nu,1} < \frac{\rho_k}{\rho_{k+1}}$$

for $\nu \geq \max\{0, \nu_0\}$, $k \in \mathbb{N}$ and for $k = 1$ and $k = 2$, we get the following inequalities

$$\frac{2(\nu+1)Q(\nu)}{3Q(\nu+2)} < r^*(h_\nu) < \frac{2Q(\nu+2)}{\frac{Q^2(\nu+2)}{(\nu+1)Q(\nu)} - \frac{3Q(\nu+4)}{4(\nu+2)}}$$

and

$$\sqrt{\frac{4(\nu+1)Q(\nu)}{\frac{Q^2(\nu+2)}{(\nu+1)Q(\nu)} - \frac{3Q(\nu+4)}{4(\nu+2)}}} < r^*(h_\nu) <$$
$$< \frac{4(\nu+3)Q(\nu)\left[-4(\nu+2)Q^2(\nu+2) + 3(\nu+1)^2 Q(\nu)Q(\nu+4)\right]}{(\nu+3)Q(\nu+2)\left[8(\nu+2)Q^2(\nu+2) - 9(\nu+1)Q(\nu)Q(\nu+4)\right] + 2(\nu+1)^2 Q^2(\nu)Q(\nu+6)}$$

and it is possible to get tighter bounds for other values of $k \in \mathbb{N}$. □

The next result concerns bounds for the radii of convexity of the functions g_ν and h_ν.

Theorem 2.4 *The following statements hold:*

a) If $\nu \geq \max\{0, \nu_0\}$ *then* $r^c(g_\nu)$ *satisfies the inequalities*

$$\frac{2}{3}\sqrt{\frac{(\nu+1)Q(\nu)}{Q(\nu+2)}} < r^c(g_\nu) < \sqrt{\frac{36(\nu+1)(\nu+2)Q(\nu+2)}{81(\nu+2)Q^2(\nu+2) - 25(\nu+1)Q(\nu)Q(\nu+4)}}$$

and

$$\sqrt[4]{\frac{16(\nu+1)^2(\nu+2)Q(\nu)}{81(\nu+2)Q^2(\nu+2)-25(\nu+1)Q(\nu)Q(\nu+4)}} < r^c(g_\nu)$$
$$< \sqrt{\frac{8(\nu+1)(\nu+3)Q(\nu)\left[81(\nu+2)Q^2(\nu+2)-25(\nu+1)Q(\nu)Q(\nu+4)\right]}{27(\nu+3)Q(\nu+2)\left[54(\nu+2)Q^2(\nu+2)-25(\nu+1)^2Q(\nu)Q(\nu+4)\right]+49(\nu+1)^2Q^2(\nu)Q(\nu+6)}}.$$

b) If $\nu \geq \max\{0, \nu_0\}$ *then* $r^c(h_\nu)$ *satisfies the inequalities*

$$\frac{(\nu+1)Q(\nu)}{Q(\nu+2)} < r^c(h_\nu) < \frac{16(\nu+2)Q(\nu)Q(\nu+2)}{16(\nu+2)Q^2(\nu+2)-9Q(\nu)Q(\nu+4)}$$

and

$$\sqrt{\frac{16(\nu+1)(\nu+2)Q^2(\nu)}{16(\nu+2)Q^2(\nu+2)-9Q(\nu)Q(\nu+4)}} < r^c(h_\nu)$$
$$< \frac{2(\nu+1)^2(\nu+3)Q(\nu)\left[16(\nu+2)Q^2(\nu+2)-9Q(\nu)Q(\nu+4)\right]}{(\nu+3)Q(\nu+2)\left[32(\nu+2)Q^2(\nu+2)-27(\nu+1)^2Q(\nu)Q(\nu+4)\right]+4(\nu+1)^2Q^2(\nu)Q(\nu+6)}.$$

Proof **a)** By using the Alexander duality theorem for starlike and convex functions we can say that the function $g_\nu(z)$ is convex if and only if $zg'_\nu(z)$ is starlike. But, the smallest positive zero of $z \mapsto z\left(zg'_\nu(z)\right)'$ is actually the radius of starlikeness of $z \mapsto \left(zg'_\nu(z)\right)$, according to Theorems 2.1 and 2.2. Therefore, the radius of convexity $r^c(g_\nu)$ is the smallest positive root of the equation $\left(zg'_\nu(z)\right)' = 0$. Therefore from (2.3), we have

$$\Delta_\nu(z) = \left(zg'_\nu(z)\right)' = \sum_{n\geq 0} \frac{(-1)^n\,(2n+1)^2 Q(2n+\nu)\Gamma(\nu+1)}{n!\Gamma(n+\nu+1)Q(\nu)}\left(\frac{z}{2}\right)^{2n}.$$

Since the function $g_\nu(z)$ belongs to the Laguerre-Pólya class of entire functions and $\mathcal{LP}$ is closed under differentiation, we can say that the function $\Delta_\nu(z) \in \mathcal{LP}$. Therefore, the zeros of the function Δ_ν are all real. Suppose that $d_{\nu,n}$ are the zeros of the function Δ_ν. Then the function Δ_ν has the infinite product representation as follows:

$$\Delta_\nu(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{d^2_{\nu,n}}\right). \tag{2.11}$$

By taking the logarithmic derivative of (2.11) we get

$$\frac{\Delta'_\nu(z)}{\Delta_\nu(z)} = -2\sum_{n\geq 1}\frac{z}{d^2_{\nu,n}-z^2} = -2\sum_{n\geq 1}\sum_{k\geq 0}\frac{1}{\left(d_{\nu,n}\right)^{2(k+1)}}z^{2k+1}$$
$$= -2\sum_{k\geq 0}\kappa_{k+1}z^{2k+1}, \qquad |z| < d_{\nu,1} \tag{2.12}$$

where $\kappa_k = \sum_{n\geq 1} \left(d_{\nu,n}\right)^{-2k}$ is the Euler-Rayleigh sum for the zeros of Δ_ν. On the other hand, by considering an infinite sum representation of $\Delta_\nu(z)$ we obtain

$$\frac{\Delta_\nu'(z)}{\Delta_\nu(z)} = \frac{\sum\limits_{n\geq 0} X_n z^{2n+1}}{\sum\limits_{n\geq 0} Y_n z^{2n}}, \tag{2.13}$$

where

$$X_n = \frac{2\,(-1)^{n+1}\,\Gamma(\nu+1)\,Q(2n+\nu+2)(2n+3)^2}{n!4^{n+1}\Gamma(n+\nu+2)\,Q(\nu)}$$

and

$$Y_n = \frac{(-1)^n\,\Gamma(\nu+1)\,Q(2n+\nu)(2n+1)^2}{n!4^n\Gamma(n+\nu+1)\,Q(\nu)}.$$

By comparing the coefficients of (2.12) and (2.13) we obtain

$$\kappa_1 = \frac{9Q(\nu+2)}{4(\nu+1)Q(\nu)},$$
$$\kappa_2 = \frac{81(\nu+2)Q^2(\nu+2) - 25(\nu+1)Q(\nu)Q(\nu+4)}{16(\nu+1)^2(\nu+2)Q(\nu)}$$

and

$$\kappa_3 = \frac{27(\nu+3)Q(\nu+2)\left[54(\nu+2)Q^2(\nu+2) - 25(\nu+1)^2Q(\nu)Q(\nu+4)\right] + 49(\nu+1)^2Q^2(\nu)Q(\nu+6)}{128(\nu+1)^3(\nu+2)(\nu+3)Q^3(\nu)}$$

By using the Euler–Rayleigh inequalities

$$\kappa_k^{-1/k} < d_{\nu,1}^2 < \frac{\kappa_k}{\kappa_{k+1}}$$

for $\nu \geq \max\{0, \nu_0\}$, $k \in \mathbb{N}$ and for $k = 1$ and $h = 2$, we get the following inequalities

$$\frac{4(\nu+1)Q(\nu)}{9Q(\nu+2)} < \left(r^c(g_\nu)\right)^2 < \frac{36(\nu+1)(\nu+2)Q(\nu)Q(\nu+2)}{81(\nu+2)Q^2(\nu+2) - 25(\nu+1)Q(\nu)Q(\nu+4)}$$

and

$$\frac{4}{9}\sqrt{\frac{(\nu+1)^2(\nu+2)Q(\nu)}{(\nu+2)Q^2(\nu+2) - 25(\nu+1)Q(\nu)Q(\nu+4)}} < \left(r^c(g_\nu)\right)^2$$
$$< \frac{8(\nu+1)(\nu+3)Q(\nu)\left[81(\nu+2)Q^2(\nu+2) - 25(\nu+1)Q(\nu)Q(\nu+4)\right]}{27(\nu+3)Q(\nu+2)\left[54(\nu+2)Q^2(\nu+2) - 25(\nu+1)^2Q(\nu)Q(\nu+4)\right] + 49(\nu+1)^2Q^2(\nu)Q(\nu+6)}$$

b) By using the same procedure as in the previous proof we can say that the radius of convexity $r^c(h_\nu)$ is the smallest positive root of the equation $\left(zh'_\nu(z)\right)' = 0$ according to Theorem 2.2. From (2.5), we have

$$\Theta_\nu(z) = \left(zh'_\nu(z)\right)' = \sum_{n\geq 0} \frac{(-1)^n\,(n+1)^2\Gamma(\nu+1)Q\,(2n+\nu)}{n!\Gamma(n+\nu+1)Q\,(\nu)} \left(\frac{z}{4}\right)^n. \quad (2.14)$$

Moreover, we know $h_\nu(z)$ belongs to the Laguerre-Pólya class of entire functions and $\mathcal{LP}$, consequently $\Theta_\nu(z) \in \mathcal{LP}$. In other words, the zeros of the function Θ_ν are all real. Assume that $l_{\nu,n}$ are the zeros of the function Θ_ν. In this case, the function Θ_ν has the infinite product representation as follows:

$$\Theta_\nu(z) = \prod_{n\geq 1}\left(1 - \frac{z}{l_{\nu,n}}\right). \quad (2.15)$$

By taking the logarithmic derivative of both sides of (2.15) for $|z| < l_{\nu,1}$ we have

$$\frac{\Theta'_\nu(z)}{\Theta_\nu(z)} = -\sum_{n\geq 1}\frac{1}{l_{\nu,n}-z} = -\sum_{n\geq 1}\sum_{k\geq 0}\frac{1}{\left(l_{\nu,n}\right)^{k+1}}z^k = -\sum_{k\geq 0}\omega_{k+1}z^k \quad (2.16)$$

where $\omega_k = \sum_{n\geq 1}\left(l_{\nu,n}\right)^{-k}$. In addition, by using the derivative of the infinite sum representation and considering the infinite sum representation of (2.14) we obtain

$$\frac{\Theta'_\nu(z)}{\Theta_\nu(z)} = \sum_{n\geq 0} T_n z^n \Big/ \sum_{n\geq 0} S_n z^n, \quad (2.17)$$

where

$$T_n = \frac{(-1)^{n+1}\,(n+2)^2\Gamma(\nu+1)Q\,(2n+\nu+2)}{n!4^{n+1}\Gamma(n+\nu+2)Q\,(\nu)}$$

and

$$S_n = \frac{(-1)^n\,(n+1)^2\Gamma(\nu+1)Q\,(2n+\nu)}{n!4^n\Gamma(n+\nu+1)Q\,(\nu)}.$$

By comparing the coefficients of (2.16) and (2.17) we get

$$\omega_1 = \frac{Q(\nu+2)}{(\nu+1)Q(\nu)},$$
$$\omega_2 = \frac{16(\nu+2)Q^2(\nu+2) - 9Q(\nu)Q(\nu+4)}{16(\nu+1)(\nu+2)Q^2(\nu)},$$
$$\omega_3 = \frac{1}{32(\nu+1)^3Q^3(\nu)}\Big(32Q^3(\nu+2)$$

$$-\frac{27(\nu+1)^2Q(\nu)Q(\nu+2)Q(\nu+4)}{(\nu+2)}+\frac{4(\nu+1)^2Q^2(\nu)Q(\nu+6)}{(\nu+2)(\nu+3)}\Bigg)$$

By using the Euler–Rayleigh inequalities

$$\omega_k^{-1/k} < l_{\nu,1} < \frac{\omega_k}{\omega_{k+1}}$$

for $\nu \geq \max\{0, \nu_0\}$, $k \in \mathbb{N}$ and for $k = 1$ and $k = 2$, we get the following inequality

$$\frac{(\nu+1)Q(\nu)}{Q(\nu+2)} < r^c(h_\nu) < \frac{16(\nu+2)Q(\nu)Q(\nu+2)}{16(\nu+2)Q^2(\nu+2)-9Q(\nu)Q(\nu+4)}$$

and

$$\sqrt{\frac{16(\nu+1)(\nu+2)Q^2(\nu)}{16(\nu+2)Q^2(\nu+2)-9Q(\nu)Q(\nu+4)}} < r^c(h_\nu)$$
$$< \frac{2(\nu+1)^2(\nu+3)Q(\nu)\left[16(\nu+2)Q^2(\nu+2)-9Q(\nu)Q(\nu+4)\right]}{(\nu+3)Q(\nu+2)\left[32(\nu+2)Q^2(\nu+2)-27(\nu+1)^2Q(\nu)Q(\nu+4)\right]+4(\nu+1)^2Q^2(\nu)Q(\nu+6)}$$

and it is possible to get tighter bounds for other values of $k \in \mathbb{N}$. □

2.3 Appendix

We know $\Gamma(z)$ is defined by

$$\frac{1}{\Gamma(z)} = z\prod_{j\geq 1}\left(1+\frac{z}{j}\right)\left(1+\frac{1}{j}\right)^{-z}, \quad z \in \mathbb{C}.$$

From 1.2 and 1.5, we have

$$\frac{2^\nu\Gamma(\nu+1)N_\nu(z^{1/2})}{Q(\nu)z^{\nu/2}} = \prod_{n\geq 1}\left(1-\frac{z}{\lambda_{\nu,n}^2}\right) = \sum_{n\geq 0}\frac{(-1)^n\,\Gamma\,(\nu+1)\,Q\,(2n+\nu)\,z^n}{n!4^n\Gamma(n+\nu+1)Q\,(\nu)}.$$

Thus, in [19] using the Lemma 1, we obtain

$$s_n = \sum_{k\geq 1}\frac{1}{\lambda_{\nu,k}^{2n}}$$

and

$$\sum_{1\leq k_1<k_2<\cdots<k_n}\frac{1}{\lambda_{\nu,k_1}^2\cdot\lambda_{\nu,k_2}^2\cdots\lambda_{\nu,k_n}^2} = \frac{\Gamma\,(\nu+1)\,Q\,(2n+\nu)}{n!4^n\Gamma(n+\nu+1)Q\,(\nu)}$$

or

$$\sum_{1\le k_1<k_2<\cdots<k_n}\frac{1}{\lambda_{\nu,k_1}^2\cdot\lambda_{\nu,k_2}^2\cdots\lambda_{\nu,k_n}^2}=\frac{Q\left(2n+\nu\right)}{n!4^n\left(\nu+1\right)_n Q\left(\nu\right)}.$$

Then, again apply the Lemma 1 in [19], with $c=-1/4$ to get

$$\sum_{k=1}^{\infty}\frac{4^n\left(-1\right)^n}{\lambda_{\nu,k}^{2n}} = \det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \frac{-Q(\nu+2)}{(\nu+1)_1 Q(\nu)} \\ \frac{Q(\nu+2)}{(\nu+1)_1 Q(\nu)} & 1 & 0 & \cdots & 0 & \frac{-Q(\nu+4)}{(\nu+1)_2 Q(\nu)} \\ \frac{Q(\nu+4)}{2!(\nu+1)_2 Q(\nu)} & \frac{Q(\nu+2)}{(\nu+1)_1 Q(\nu)} & 1 & \cdots & 0 & \frac{-Q(\nu+6)}{2!(\nu+1)_3 Q(\nu)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{Q(\nu+2(n-2))}{(n-2)!(\nu+1)_{n-2} Q(\nu)} & \frac{Q(\nu+2(n-3))}{(n-3)!(\nu+1)_{n-3} Q(\nu)} & \frac{Q(\nu+2(n-4))}{(n-4)!(\nu+1)_{n-4} Q(\nu)} & \cdots & 1 & \frac{-Q(\nu+2(n-1))}{(n-2)!(\nu+1)_{n-1} Q(\nu)} \\ \frac{Q(\nu+2(n-1))}{(n-1)!(\nu+1)_{n-1} Q(\nu)} & \frac{Q(\nu+2(n-2))}{(n-2)!(\nu+1)_{n-2} Q(\nu)} & \frac{Q(\nu+2(n-3))}{(n-3)!(\nu+1)_{n-3} Q(\nu)} & \cdots & \frac{Q(\nu+2)}{(\nu+1)_1 Q(\nu)} & \frac{-Q(\nu+2n)}{(n-1)!(\nu+1)_n Q(\nu)} \end{pmatrix}.$$

The first few s_n are as follows:

$$s_1=\frac{Q\left(\nu+2\right)}{4\left(\nu+1\right)_1 Q\left(\nu\right)}$$

$$s_2=\frac{\left(\nu+2\right)Q^2\left(\nu+2\right)-\left(\nu+1\right)Q\left(\nu\right)Q\left(\nu+4\right)}{4^2Q^2\left(\nu\right)\prod_{j=1}^{2}\left(\nu+1\right)_j}$$

$$s_3=\frac{\begin{bmatrix}\left(\nu+2\right)_2 Q\left(\nu+2\right)\left[2\left(\nu+2\right)Q^2\left(\nu+2\right)-3\left(\nu+1\right)Q\left(\nu\right)Q\left(\nu+4\right)\right]\\ +\left(\nu+1\right)^2\left(\nu+2\right)Q^2\left(\nu\right)Q\left(\nu+6\right)\end{bmatrix}}{2!4^3Q^3\left(\nu\right)\prod_{j=1}^{3}\left(\nu+1\right)_j}$$

$$s_4=\frac{\begin{bmatrix}\left(\nu+2\right)_2\left[6\left(\nu+2\right)\left(\nu+2\right)_3 Q^4\left(\nu+2\right)-12\left(\nu+1\right)_4 Q\left(\nu\right)Q^2\left(\nu+2\right)Q\left(\nu+4\right)\right.\\ +\left(\nu+1\right)\left(\nu+1\right)_2\left(\nu+4\right)Q^2\left(\nu\right)Q\left(\nu+2\right)\left(Q\left(\nu+4\right)+3Q\left(\nu+6\right)\right)\\ \left.+\left(\nu+1\right)^2Q^2\left(\nu\right)\left(3\left(\nu+2\right)_2Q^2\left(\nu+4\right)-\left(\nu+1\right)_2Q\left(\nu\right)Q\left(\nu+8\right)\right)\right]\end{bmatrix}}{3!4^4Q^4\left(\nu\right)\prod_{j=1}^{4}\left(\nu+1\right)_j}.$$

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