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RADII OF STARLIKENESS AND CONVEXITY OF THE DERIVATIVES OF BESSEL FUNCTION

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We find the radii of starlikeness and convexity of the derivatives of Bessel function for three different kinds of normalization. The key tools in the proof of our main results are the Mittag-Leffler expansion for the nth derivative of the Bessel function and the properties of its real zeros. In addition, by using the Euler–Rayleigh inequalities we obtain some tight lower and upper bounds for the radii of starlikeness and convexity of order zero for the normalized nth derivative of the Bessel function. As the main results of our investigations, we can mention natural extensions of some known results for the classical Bessel functions of the first kind.

1. Introduction

We denote by $\mathbb{D}_r = \{z \in \mathbb{C} \colon |z| < r\}$ $(r > 0)$ a disk of radius r. Also let $\mathbb{D} = \mathbb{D}_1$ and let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D}$ satisfying the ordinary normal conditions

$$f(0) = f'(0) - 1 = 0.$$

Traditionally, the subclass of $\mathcal{A}$ formed by univalent functions is denoted by $\mathcal{S}$. We say that the function $f \in \mathcal{A}$ is starlike in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin. Analytically, the function f is starlike in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}_r.$$

For $\beta \in [0, 1)$, we say that f is a starlike function of order β in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{D}_r.$$

By a real number

$$r^*_\beta(f) = \sup\left\{r \in (0, r_f) \colon \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \ \text{ for all } \ z \in \mathbb{D}_r\right\}$$

we define the radius of starlikeness of order β for a function f. Note that $r^*(f) = r^*_0(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^*_\beta(f)})$ is a starlike domain with respect to the origin.

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A function $f \in \mathcal{A}$ is convex in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$ and $f(\mathbb{D}_r)$ is a convex domain in $\mathbb{C}$. Analytically, the function f is convex in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}_r.$$

For $\beta \in [0,1)$, we say that f is a convex function of order β in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta, \quad z \in \mathbb{D}_r.$$

The radius of convexity of the order β of the function f is defined as the real number

$$r^c_\beta(f) = \sup\left\{r \in (0, r_f) \colon \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta \text{ for all } z \in \mathbb{D}_r\right\}.$$

Note that $r^c(f) = r^c_0(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^c_\beta(f)})$ is a convex domain.

The Bessel function of the first kind of order ν is defined as follows [18, p. 217]:

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \quad z \in \mathbb{C}.$$

We now consider the nth derivative of the Bessel function of the first kind:

$$J^{(n)}_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2m+\nu+1)}{m! 2^n \Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m-n+\nu}, \quad z \in \mathbb{C}.$$

Here, it is worth noting that, for $n = 0$, $J^{(n)}_\nu$ reduces to the classical Bessel function J_ν. Since the function $J^{(n)}_\nu$ does not belong to $\mathcal{A}$, we first introduce and consider the following normalized forms:

$$\begin{aligned} f_{\nu,n}(z) &= \left[2^\nu \Gamma(\nu-n+1) J^{(n)}_\nu(z)\right]^{\frac{1}{\nu-n}}, \\ g_{\nu,n}(z) &= 2^\nu \Gamma(\nu-n+1) z^{1+n-\nu} J^{(n)}_\nu(z), \\ h_{\nu,n}(z) &= 2^\nu \Gamma(\nu-n+1) z^{1+\frac{n-\nu}{2}} J^{(n)}_\nu(\sqrt{z}), \end{aligned} \tag{1.1}$$

where $\nu > n-1$.

The first studies of the geometric properties of Bessel functions of the first kind were carried out in 1960 by Brown, Kreyszig, and Todd [10, 16]. They determined the radius of starlikeness of the functions $f_{\nu,0}(z)$ and $g_{\nu,0}(z)$ for the case $\nu > 0$. Recently, in 2014, Baricz, et al. [3] and Baricz and Szász [4] have obtained, respectively, the radius of starlikeness of order β and the radius of convexity of order β for the functions $f_{\nu,0}(z)$, $g_{\nu,0}(z)$, and $h_{\nu,0}(z)$ in the case where $\nu > -1$. On the other hand, we know that if $\nu \in (-2,-1)$, then the Bessel function has exactly two pure imaginary conjugate complex zeros, whereas all other zeros are real [21, p. 483]. In 2015, Szász [20] investigated the radius of starlikeness of order β for the functions $g_\nu(z)$ and $h_\nu(z)$ in the case where $\nu \in (-2,-1)$ by using certain inequalities. In the same year, Baricz and Szász [5] obtained the radius of convexity of order β for the functions $g_\nu(z)$ and $h_\nu(z)$ in the case where $\nu \in (-2,-1)$. Later, in 2016, Baricz et al. [7] determined the radius of α-convexity of the same three functions for $\nu > -1$. In a year, Çağlar, et al. [11] extended this result to the case where $\nu \in (-2,-1)$. In 2017, Deniz and Szász [12] determined the radius of uniform convexity of $f_{\nu,0}(z)$, $g_{\nu,0}(z)$, and $h_{\nu,0}(z)$ for $\nu > -1$. They also established necessary and sufficient conditions for the parameters of these three normalized functions under which these functions are uniformly convex in the unit disk. Moreover, in [1, 2], the authors established tight lower and upper bounds for the radii of starlikeness and convexity of the functions $g_{\nu,0}(z)$ and $h_{\nu,0}(z)$. The key tools in their proofs were some new Mittag-Leffler expansions for the quotients of Bessel functions of the first kind, special properties of the zeros of Bessel functions of the first kind and their derivatives, Euler–Rayleigh inequalities, and the fact that the smallest positive zeros of some Dini functions are smaller than the first positive zero of Bessel function of the first kind.

Another study of Bessel functions deal with the properties of the derivatives and zeros of these derivatives. In the last three decades, the zeros of the nth derivative of Bessel functions of the first kind for $n \in \{1,2,3\}$ have been also studied by numerous researchers, including Elbert, Ifantis, Ismail, Kokologiannaki, Laforgia, Landau, Lorch, Mercer, Muldoon, Petropoulou, Siafarikas, and Szegö; for more details see the papers [13, 15] and the references therein. Very recently, in 2018, Baricz, et al. [8] have obtained some results for the zeros of the nth derivative of Bessel functions of the first kind for all $n \in \mathbb{N}$ by using the Laguerre–Pólya class of entire functions and the so-called Laguerre inequalities.

Motivated by the results outlined above, in the preset paper, we deal with the radii of starlikeness and convexity of order β for the functions $f_{\nu,n}(z)$, $g_{\nu,n}(z)$, and $h_{\nu,n}(z)$ in the case where $\nu > n-1$ for $n \in \mathbb{N}$. In addition, we establish tight lower and upper bounds for the radii of starlikeness and convexity of these functions.

2. Preliminaries

In order to prove our main results, we need the following preliminary results.

Lemma 2.1 [8]. *The following assertions are valid:*

(a) If $\nu > n-1$, then $z \mapsto J_\nu^{(n)}(z)$ has infinitely many zeros, which are all real and simple, except the origin.

(b) If $\nu > n$, then the positive zeros of the nth and $(n+1)$th derivative of J_ν are interlacing.

(c) If $\nu > n-1$, then all zeros of $z \mapsto (n-\nu)J_\nu^{(n)}(z) + zJ_\nu^{(n+1)}(z)$ are real and interlace with the zeros of $z \mapsto J_\nu^{(n)}(z)$.

The next lemma (see [9, 19]) is also required for our presentation.

Lemma 2.2. *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R}, \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n, \quad b_n > 0,$$

for all $n \geq 0$, converge on the interval $(-r,r)$ for some $r > 0$. If the sequence $\{a_n/b_n\}_{n\geq 0}$ is decreasing (increasing), then the function $x \to f(x)/g(x)$ is also decreasing (increasing) on $(0,r)$. Hence, the same result holds for

$$f(x) = \sum_{n=0}^{\infty} a_n x^{2n} \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} b_n x^{2n}.$$

2.1. Zeros of Hyperbolic Polynomials and the Laguerre–Pólya Class of Entire Functions. In this section, we present some necessary knowledge about polynomials and entire functions with real zeros. An algebraic polynomial is named hyperbolic if its all zeros are real. In what follows, we use the following lemma given in [6] and obtain new results:

Lemma 2.3. *Assume that*

$$p(x) = 1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots + (-1)^n a_n x^n = (1 - x/x_1) \ldots (1 - x/x_n)$$

is a hyperbolic polynomial with positive zeros $0 < x_1 \leq x_2 \leq \ldots \leq x_n,$ *and that it is normalized by* $p(0) = 1.$ *Then the polynomial* $q(x) = Cp(x) - xp'(x)$ *is hyperbolic for any constant* $C.$ *Moreover, the smallest zero* η_1 *is in* $(0, x_1)$ *if and only if* $C < 0.$

Clearly, a real entire function ψ is in the Laguerre–Pólya class $\mathcal{LP}$ if it has the form

$$\psi(x) = cx^m e^{-ax^2+\beta x} \prod_{k \geq 1} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

with $c, \beta, x_k \in \mathbb{R},\ a \geq 0,\ m \in \mathbb{N} \cup \{0\},$ and $\sum x_k^{-2} < \infty.$ Similarly, we say that ϕ is of type $\mathcal{I}$ in the Laguerre–Pólya class denoted by $\phi \in \mathcal{LPI}$ if $\phi(x)$ or $\phi(-x)$ can be represented as

$$\phi(x) = cx^m e^{\sigma x} \prod_{k \geq 1} \left(1 + \frac{x}{x_k}\right),$$

with $c \in \mathbb{R},\ \sigma \geq 0,\ m \in \mathbb{N} \cup \{0\},\ x_k > 0,$ and $\sum x_k^{-1} < \infty.$ The complement of the space of hyperbolic polynomials in the topology induced by the uniform convergence on compact sets of the complex plane is the class $\mathcal{LP}$ if the complement of the hyperbolic polynomials whose zeros possess a preassigned constant sign is $\mathcal{LPI}.$ For any entire function φ of the form

$$\varphi(x) = \sum_{k \geq 0} \mu_k \frac{x^k}{k!},$$

its Jensen polynomials are given by

$$P_m(\varphi; x) = P_m(x) = \sum_{k=0}^{m} \binom{m}{k} \mu_k x^k.$$

The following lemma is a well-known characterization of functions from the class $\mathcal{LP}$ (see [14]):

Lemma 2.4. φ *is in the class* $\mathcal{LP}$ *(*$\mathcal{LPI}$*, respectively) if and only if all polynomials* $P_m(\varphi; x),\ m = 1, 2, \ldots,$ *are hyperbolic and such that they are hyperbolic with zeros of the same sign. Moreover, the sequence* $P_m(\varphi; z/n)$ *is locally uniformly convergent to* $\varphi(z).$

The following lemma is necessary for the proof of our main results:

Lemma 2.5. *Let* $\nu > n - 1$ *and* $a < 0.$ *Then the functions*

$$z \longmapsto (2a - n + \nu) J_\nu^{(n)}(z) - z J_\nu^{(n+1)}(z)$$

can be written in the form

$$2^{n-1}\Gamma(\nu+1-n)\Big((2a-n+\nu)J_\nu^{(n)}(z)-zJ_\nu^{(n+1)}(z)\Big)=\left(\frac{z}{2}\right)^{\nu-n}W_{\nu,n}(z),$$

where $W_{\nu,n}$ are entire functions from the Laguerre–Pólya class $\mathcal{LP}$. Moreover, the smallest positive zero of $W_{\nu,n}$ cannot exceed the first positive zero $j_{\nu,1}^{(n)}$, where $j_{\nu,m}^{(n)}$ is the mth positive zero of $J_\nu^{(n)}(z)$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$.

Proof. It is obvious from the infinite product representation of

$$z\longmapsto\mathcal{J}_\nu^{(n)}(z)=2^\nu\Gamma(\nu+1-n)(z)^{n-\nu}J_\nu^{(n)}(z)$$

that this function belongs to the class $\mathcal{LP}$. This means that the function

$$z\longmapsto\mathbb{J}_\nu^{(n)}(z)=\mathcal{J}_\nu^{(n)}(2\sqrt{z}\,)$$

is in the class $\mathcal{LPI}$. Thus, by Lemma 2.4, all its Jensen polynomials

$$P_m(\mathbb{J}_\nu^{(n)};\varsigma)=\sum_{k=0}^{m}\binom{m}{k}\mu_k x^k$$

are hyperbolic. However, it can be seen that the Jensen polynomials of $\widetilde{W}_{\nu,n}(z)=W_{\nu,n}(2\sqrt{z}\,)$ are, clearly,

$$P_m\big(\widetilde{W}_{\nu,n};\varsigma\big)=aP_m\big(\mathbb{J}_\nu^{(n)};\varsigma\big)-\varsigma P_m'\big(\mathbb{J}_\nu^{(n)};\varsigma\big).$$

Moreover, Lemma 2.3 tells us that all zeros of $P_m\big(\widetilde{W}_{\nu,n};\varsigma\big)$ are real and positive and that the smallest zero precedes the first zero of $P_m\big(\mathbb{J}_\nu^{(n)};\varsigma\big)$. By Lemma 2.4, the last result immediately implies that $\widetilde{W}_{\nu,n}\in\mathcal{LPI}$ and that its first zero precedes $j_{\nu,1}^{(n)}$. Finally, the first part of the statement of the lemma follows if we go back from $\widetilde{W}_{\nu,n}$ to $W_{\nu,n}$ by setting $\varsigma=\dfrac{z^2}{4}$.

Lemma 2.5 is proved.

2.2. Euler–Rayleigh Sums for Positive Zeros of $J_\nu^{(n)}(z)$. Baricz, et al. [8] proved the Weierstrassian decomposition of $J_\nu^{(n)}(z)$ as follows:

$$J_\nu^{(n)}(z)=\frac{z^{\nu-n}}{2^\nu\Gamma(\nu+1-n)}\prod_{m\geq1}\left(1-\frac{z^2}{\left(j_{\nu,m}^{(n)}\right)^2}\right),\tag{2.1}$$

where $j_{\nu,m}^{(n)}$ is the mth positive zero of $J_\nu^{(n)}(z)$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$. Therefore, we can write

$$g_{\nu,n}(z)=2^\nu\Gamma(\nu-n+1)z^{1+n-\nu}J_\nu^{(n)}(z)=z\prod_{m\geq1}\left(1-\frac{z^2}{\left(j_{\nu,m}^{(n)}\right)^2}\right).\tag{2.2}$$

On the other hand, the series representation of $g_{\nu,n}(z)$ is

$$g_{\nu,n}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m! 4^m \Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} z^{2m+1}. \tag{2.3}$$

We would now like to mention that, by using equations (2.2) and (2.3), we can get the following Euler–Rayleigh sums for the positive zeros of the function $g_{\nu,n}$. It follows from equality (2.3) that

$$g_{\nu,n}(z) = z - \frac{\nu+2}{4(\nu-n+2)(\nu-n+1)} z^3 + \frac{(\nu+4)(\nu+3)}{32(\nu-n+4)(\nu-n+3)(\nu-n+2)(\nu-n+1)} z^5 - \ldots. \tag{2.4}$$

Further, if we consider (2.2), then some calculations imply that

$$g_{\nu,n}(z) = z - \sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^2} z^3 + \frac{1}{2}\left(\left(\sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^2}\right)^2 - \sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^4}\right) z^5 - \ldots. \tag{2.5}$$

By equating the first few coefficients with the same degrees in equations (2.4) and (2.5), we get

$$\sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^2} = \frac{\nu+2}{4(\nu-n+2)(\nu-n+1)} \tag{2.6}$$

and

$$\sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^4} = \frac{1}{16(\nu-n+2)(\nu-n+1)} \times \left(\frac{(\nu+2)^2}{(\nu-n+2)(\nu-n+1)} - \frac{(\nu+4)(\nu+3)}{(\nu-n+4)(\nu-n+3)}\right). \tag{2.7}$$

Here, it is worth noting that, for $n=0,$ equations (2.6) and (2.7) are reduced to

$$\sum_{m\geq 1} \frac{1}{(j_{\nu,m})^2} = \frac{1}{4(\nu+1)} \quad \text{and} \quad \sum_{m\geq 1} \frac{1}{(j_{\nu,m})^4} = \frac{1}{16(\nu+2)(\nu+1)^2},$$

respectively, where $j_{\nu,m}$ denotes the mth zero of the classical Bessel function J_ν.

In another special case, for $n=1,2,$ equations (2.6) and (2.7) are reduced to

$$\sum_{m\geq 1} \frac{1}{(j'_{\nu,m})^2} = \frac{\nu+2}{4\nu(\nu+1)}, \qquad \sum_{m\geq 1} \frac{1}{(j'_{\nu,m})^4} = \frac{\nu^2+8\nu+8}{16\nu^2(\nu+1)^2(\nu+2)}$$

and

$$\sum_{m\geq 1}\frac{1}{\left(j''_{\nu,m}\right)^2}=\frac{\nu+2}{4(\nu-1)\nu},\qquad \sum_{m\geq 1}\frac{1}{\left(j''_{\nu,m}\right)^4}=\frac{13\nu^3+19\nu^2+26\nu+8}{16(\nu-1)^2\nu^2(\nu+1)(\nu+2)},$$

where $j'_{\nu,m}$ and $j''_{\nu,m}$ denote the mth zeros of the functions J'_ν and J''_ν, respectively.

3. Main Results

3.1. Radii of Starlikeness and Convexity for the Functions $f_{\nu,n}$, $g_{\nu,n}$ and $h_{\nu,n}$. The first principal established result concerns the radii of starlikeness and can be formulated as follows. Here and in what follows, I_ν denotes the modified Bessel function of the first kind and order ν. Note that $I_\nu(z)=i^{-\nu}J_\nu(iz)$.

Theorem 3.1. *The followings assertions are true:*

(a) If $\nu>n$ and $\beta\in[0,1)$, then

$$r^*_\beta(f_{\nu,n})=x^{(n)}_{\nu,1},$$

where $x^{(n)}_{\nu,1}$ is the smallest positive root of the equation $\dfrac{rJ^{(n+1)}_\nu(r)}{(\nu-n)J^{(n)}_\nu(r)}-\beta=0.$

In addition, if $n-1<\nu<n$ and $\beta\in[0,1)$, then

$$r^*_\beta(f_{\nu,n})=x^{(n)}_{\nu,2},$$

where $x^{(n)}_{\nu,2}$ is the smallest positive root of the equation $\dfrac{rI^{(n+1)}_\nu(r)}{(\nu-n)I^{(n)}_\nu(r)}-\beta=0.$

(b) If $\nu>n-1$ and $\beta\in[0,1)$, then

$$r^*_\beta(g_{\nu,n})=y^{(n)}_{\nu,1},$$

where $y^{(n)}_{\nu,1}$ is the smallest positive root of the equation $\dfrac{rJ^{(n+1)}_\nu(r)}{J^{(n)}_\nu(r)}+n+1-\nu-\beta=0.$

(c) If $\nu>n-1$ and $\beta\in[0,1)$, then

$$r^*_\beta(h_{\nu,n})=z^{(n)}_{\nu,1},$$

where $z^{(n)}_{\nu,1}$ is the smallest positive root of the equation $\dfrac{\sqrt{r}J^{(n+1)}_\nu(\sqrt{r})}{J^{(n)}_\nu(\sqrt{r})}+n+2-\nu-2\beta=0.$

Proof. First, we prove part (a) for $\nu>n$ and parts (b) and (c) for $\nu>n-1$. It is necessary to show that the following inequalities:

$$\operatorname{Re}\left(\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)}\right)>\beta,\qquad \operatorname{Re}\left(\frac{zg'_{\nu,n}(z)}{g_{\nu,n}(z)}\right)>\beta\qquad\text{and}\qquad \operatorname{Re}\left(\frac{zh'_{\nu,n}(z)}{h_{\nu,n}(z)}\right)>\beta \tag{3.1}$$

are valid for $z \in \mathbb{D}_{r^*_\beta(f_{\nu,n})}$, $z \in \mathbb{D}_{r^*_\beta(g_{\nu,n})}$ and $z \in \mathbb{D}_{r^*_\beta(h_{\nu,n})}$, respectively, and each inequality presented above cannot hold in larger disks.

If we write equation (2.1) in the definitions of the functions $f_{\nu,n}(z)$, $g_{\nu,n}(z)$, and $h_{\nu,n}(z)$, then, by using the logarithmic derivation, we get

$$\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)} = \frac{1}{\nu-n}\frac{zJ_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)} = 1 - \frac{1}{\nu-n}\sum_{m\geq 1}\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}, \quad \nu > n,$$

$$\frac{zg'_{\nu,n}(z)}{g_{\nu,n}(z)} = n+1-\nu + \frac{zJ_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)} = 1 - \sum_{m\geq 1}\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}, \quad \nu > n-1,$$

$$\frac{zh'_{\nu,n}(z)}{h_{\nu,n}(z)} = 1 + \frac{n-\nu}{2} + \frac{1}{2}\frac{\sqrt{z}J_\nu^{(n+1)}(\sqrt{z})}{J_\nu^{(n)}(\sqrt{z})} = 1 - \sum_{m\geq 1}\frac{z}{\left(j_{\nu,m}^{(n)}\right)^2 - z}, \quad \nu > n-1.$$

It is known [4] that if $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are such that $\lambda > |z|$, then

$$\frac{|z|}{\lambda - |z|} \geq \operatorname{Re}\left(\frac{z}{\lambda - z}\right). \tag{3.2}$$

Thus, the inequality

$$\frac{|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 - |z|^2} \geq \operatorname{Re}\left(\frac{z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right)$$

holds for every $\nu > n-1$. Therefore,

$$\operatorname{Re}\left(\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)}\right) = 1 - \frac{1}{\nu-n}\sum_{m\geq 1}\operatorname{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right)$$

$$\geq 1 - \frac{1}{\nu-n}\sum_{m\geq 1}\frac{2|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 - |z|^2} = \frac{|z|f'_{\nu,n}(|z|)}{f_{\nu,n}(|z|)},$$

$$\operatorname{Re}\left(\frac{zg'_{\nu,n}(z)}{g_{\nu,n}(z)}\right) = 1 - \sum\nolimits_{m\geq 1}\operatorname{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right)$$

$$\geq 1 - \sum_{m\geq 1}\frac{2|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 - |z|^2} = \frac{|z|g'_{\nu,n}(|z|)}{g_{\nu,n}(|z|)},$$

$$\operatorname{Re}\left(\frac{zh'_{\nu,n}(z)}{h_{\nu,n}(z)}\right) = 1 - \sum_{m\geq 1} \operatorname{Re}\left(\frac{z}{\left(j_{\nu,m}^{(n)}\right)^2 - z}\right)$$

$$\geq 1 - \sum_{m\geq 1} \frac{|z|}{\left(j_{\nu,m}^{(n)}\right)^2 - |z|} = \frac{|z|h'_{\nu,n}(|z|)}{h_{\nu,n}(|z|)},$$

where the equalities are attained only for $z = |z| = r.$ By using the last inequalities and the minimum principle for harmonic functions, we conclude that the corresponding inequalities in (3.1) hold if and only if

$$|z| < x_{\nu,1}^{(n)}, \qquad |z| < y_{\nu,1}^{(n)}, \qquad \text{and} \qquad |z| < z_{\nu,1}^{(n)},$$

respectively, where $x_{\nu,1}^{(n)},\ y_{\nu,1}^{(n)}$, and $z_{\nu,1}^{(n)}$ are the smallest positive roots of the equations

$$\frac{rf'_{\nu,n}(r)}{f_{\nu,n}(r)} = \beta, \qquad \frac{rg'_{\nu,n}(r)}{g_{\nu,n}(r)} = \beta, \qquad \text{and} \qquad \frac{rh'_{\nu,n}(r)}{h_{\nu,n}(r)} = \beta,$$

respectively, which are equivalent to

$$\frac{rJ_\nu^{(n+1)}(r)}{(\nu - n)J_\nu^{(n)}(r)} - \beta = 0, \qquad \frac{rJ_\nu^{(n+1)}(r)}{J_\nu^{(n)}(r)} + n + 1 - \nu - \beta = 0,$$

and

$$\frac{\sqrt{r}J_\nu^{(n+1)}(\sqrt{r})}{J_\nu^{(n)}(\sqrt{r})} + n + 2 - \nu - 2\beta = 0.$$

The result follows from Lemma 2.5 if, instead of $a,$ we take the values

$$\frac{(\beta-1)(\nu-n)}{2}, \qquad \frac{\beta-1}{2}, \qquad \text{and} \qquad \beta - 1,$$

respectively. In other words, Lemma 2.5 shows that all zeros of the indicated three functions are real and their first positive zeros do not exceed the first positive zeros $j_{v,1}^{(n)}$ and $\sqrt{j_{v,1}^{(n)}}$. This guarantees that the above inequalities hold. This completes the proof of part (a) for $\nu > n$ and parts (b) and (c) for $\nu > n - 1.$

To prove the statement of part (a) for $\nu \in (n-1, n),$ we use the counterpart of (3.2), i.e.,

$$\operatorname{Re}\left(\frac{z}{\lambda - z}\right) \geq \frac{-|z|}{\lambda + |z|}, \tag{3.3}$$

which holds for all $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ such that $\lambda > |z|$ (see [3]). If, in inequality (3.3), we replace z by z^2 and λ by $\left(j_{\nu,m}^{(n)}\right)^2$, then we get

$$\operatorname{Re}\left(\frac{z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right) \geq \frac{-|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 + |z|^2}$$

provided that $|z| < j_{\nu,1}^{(n)}.$

Thus, for $n-1<\nu<n,$ we obtain

$$\operatorname{Re}\left(\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)}\right)=1-\frac{1}{\nu-n}\sum_{m\geq 1}\operatorname{Re}\left(\frac{2z^2}{\left(j^{(n)}_{\nu,m}\right)^2-z^2}\right)$$

$$\geq 1+\frac{1}{\nu-n}\sum_{m\geq 1}\frac{2|z|^2}{\left(j^{(n)}_{\nu,m}\right)^2+|z|^2}=\frac{i|z|f'_{\nu,n}(i|z|)}{f_{\nu,n}(i|z|)}.$$

In this case, equality is attained for $z=i|z|=ir.$ Moreover, the last inequality implies that

$$\operatorname{Re}\left(\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)}\right)>\beta$$

if and only if $|z|<x^{(n)}_{v,2},$ where $x^{(n)}_{v,2}$ denotes the smallest positive root of the equations $\dfrac{irf'_{\nu,n}(ir)}{f_{\nu,n}(ir)}=\beta,$ which is equivalent to

$$\frac{irJ^{(n+1)}_{\nu}(ir)}{(\nu-n)J^{(n)}_{\nu}(ir)}-\beta=0 \qquad \text{or} \qquad \frac{rI^{(n+1)}_{\nu}(r)}{(\nu-n)I^{(n)}_{\nu}(r)}-\beta=0$$

for $n-1<\nu<n.$ It follows from Lemma 2.5 that the first positive zero of

$$z\mapsto irJ^{(n+1)}_{\nu}(ir)-\beta(\nu-n)J^{(n)}_{\nu}(ir)$$

cannot exceed $j^{(n)}_{\nu,1}$ and, hence, the above inequalities are verified. Thus, it remains to show that the function presented above has actually only one zero in $(0,\infty).$ Note that, by virtue of Lemma 2.2, the function

$$r\mapsto \frac{irJ^{(n+1)}_{\nu}(ir)}{J^{(n)}_{\nu}(ir)}=\frac{Q_1}{Q_2},$$

where

$$Q_1=\sum_{m=0}^{\infty}\frac{(2m-n+\nu)\Gamma(2m+\nu+1)}{m!2^{2m+\nu}\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}r^{2m},$$

$$Q_2=\sum_{m=0}^{\infty}\frac{\Gamma(2m+\nu+1)}{m!2^{2m+\nu}\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}r^{2m},$$

is increasing on $(0,\infty)$ as a quotient of two power series whose positive coefficients form an increasing "quotient sequence" $\{2m-n+\nu\}_{m\geq 0}.$ On the other hand, the above function tends to $\nu-n$ as $r\to 0$ and, hence, its graph may cross the horizontal line $y=\beta(\nu-n)>\nu-n$ only once. Thus, the proof of part (a) of the theorem is completed for $\nu\in(n-1,n).$

Theorem 3.1 is proved.

Table 3.1

n	$r^*_\beta(f_{2.5,n})$		$r^*_\beta(g_{2.5,n})$		$r^*_\beta(h_{2.5,n})$	
	$\beta=0$	$\beta=0.5$	$\beta=0$	$\beta=0.5$	$\beta=0$	$\beta=0.5$
0	3.6328	2.7569	2.5011	1.8192	11.1696	6.2556
1	2.1056	1.5926	1.7975	1.3307	5.4265	3.2312
2	0.8512	0.6229	1.1285	0.8512	2.0284	1.2735
3	0.4586	0.3051	0.4819	0.3703	0.3543	0.2323

With regard for Theorem 3.1, we tabulate the radius of starlikeness for $f_{\nu,n},\ g_{\nu,n}$ and $h_{\nu,n}$ for fixed $\nu=2.5$, $n=0,1,2,3$, and, respectively, $\beta=0$ and $\beta=0.5$. These values are presented in Table 3.1. Moreover, in Table 3.1, we see that the radius of starlikeness is decreasing according to the order of the derivative and the order of starlikeness. In other words, all these results enable us to conclude that

$$r^*_\beta(f_{\nu,0}) > r^*_\beta(f_{\nu,1}) > r^*_\beta(f_{\nu,2}) > \ldots > r^*_\beta(f_{\nu,n}) > \ldots$$

for $\beta\in[0,1)$ and $\nu>n-1,\ n\in\mathbb{N}_0$. In addition, we can write

$$r^*_{\beta_1}(f_{\nu,n}) < r^*_{\beta_0}(f_{\nu,n})$$

for $0\le\beta_0<\beta_1<1$ and $\nu>n-1,\ n\in\mathbb{N}_0$. The same inequalities are also true for $r^*_\beta(g_{\nu,n})$ and $r^*_\beta(h_{\nu,n})$.

For $n=0$ in Theorem 3.1, we obtain the results of Baricz, et al. [3]. Our results is a common generalization of these results.

The second established principal result concerns the radii of convexity and can be formulated as follows:

Theorem 3.2. *The following statements hold:*

(a) If $\nu>n$ and $\beta\in[0,1)$, then the radius $r^c_\beta(f_{\nu,n})$ is the smallest positive root of the equation

$$1-\beta+\frac{rJ_\nu^{(n+2)}(r)}{J_\nu^{(n+1)}(r)}+\left(\frac{1}{\nu-n}-1\right)\frac{rJ_\nu^{(n+1)}(r)}{J_\nu^{(n)}(r)}=0.$$

Moreover,

$$r^c_\beta(f_{\nu,n}) < j^{(n+1)}_{\nu,1} < j^{(n)}_{\nu,1}.$$

(b) If $\nu>n-1$ and $\beta\in[0,1)$, then the radius $r^c_\beta(g_{\nu,n})$ is the smallest positive root of the equation

$$n+1-\nu-\beta+\frac{(n-\nu+2)rJ_\nu^{(n+1)}(r)+r^2J_\nu^{(n+2)}(r)}{(n-\nu+1)J_\nu^{(n)}(r)+rJ_\nu^{(n+1)}(r)}=0.$$

(c) If $\nu > n-1$ and $\beta \in [0,1)$, then the radius $r^c_\beta(h_{\nu,n})$ is the smallest positive root of the equation

$$\frac{n+2-\nu-2\beta}{2}+\frac{\sqrt{r}}{2}\frac{(n-\nu+3)J_\nu^{(n+1)}(\sqrt{r})+\sqrt{r}J_\nu^{(n+2)}(\sqrt{r})}{(n-\nu+2)J_\nu^{(n)}(\sqrt{r})+\sqrt{r}J_\nu^{(n+1)}(\sqrt{r})}=0.$$

Proof. (a) Since

$$1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}=1+\frac{zJ_\nu^{(n+2)}(z)}{J_\nu^{(n+1)}(z)}+\left(\frac{1}{\nu-n}-1\right)\frac{zJ_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)}$$

and, in view of (2.1),

$$\frac{zJ_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)}=\nu-n-\sum_{m\geq 1}\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2-z^2},$$

we conclude that

$$1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}=1-\left(\frac{1}{\nu-n}-1\right)\sum_{m\geq 1}\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2-z^2}-\sum_{m\geq 1}\frac{2z^2}{\left(j_{\nu,m}^{(n+1)}\right)^2-z^2}.$$

We now suppose that $\nu \in (n, n+1]$. If we use inequality (3.2) for all $z \in \mathbb{D}_{j_{\nu,1}^{(n)}}$, then we get the inequality

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}\right)&=1-\left(\frac{1}{\nu-n}-1\right)\sum_{m\geq 1}\mathrm{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2-z^2}\right)\\&\quad-\sum_{m\geq 1}\mathrm{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n+1)}\right)^2-z^2}\right)\\&\geq 1-\left(\frac{1}{\nu-n}-1\right)\sum_{m\geq 1}\frac{2r^2}{\left(j_{\nu,m}^{(n)}\right)^2-r^2}-\sum_{m\geq 1}\frac{2r^2}{\left(j_{\nu,m}^{(n+1)}\right)^2-r^2},\end{aligned}$$

where $|z|=r$. Moreover, we conclude that if we use the inequality [4] (Lemma 2.1)

$$\mu\mathrm{Re}\left(\frac{z}{a-z}\right)-\mathrm{Re}\left(\frac{z}{b-z}\right)\geq \mu\frac{|z|}{a-|z|}-\frac{|z|}{b-|z|},$$

where $a > b > 0$, $\mu \in [0,1]$, and $z \in \mathbb{C}$ such that $|z| < b$, then we see that the above inequality is also true for $\nu > n+1$. Here, we have used the fact that the zeros of the nth and $(n+1)$th derivatives of J_v are interlacing according to Lemma 2.1. For $r \in (0, j_{\nu,1}^{(n)})$, the above inequality implies that

$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}\right)\right\}=1+\frac{rf''_{\nu,n}(r)}{f'_{\nu,n}(r)}.$$

On the other hand, we define the function $\varphi_{\nu,n}\colon (n, j_{\nu,1}^{(n)}) \to \mathbb{R}$ as follows:

$$\varphi_{\nu,n}(r) = 1 + \frac{r f''_{\nu,n}(r)}{f'_{\nu,n}(r)}.$$

Since the zeros of the nth and $(n+1)$th derivatives of J_v are interlacing according to Lemma 2.1 and

$$r < j_{\nu,1}^{(n+1)} < j_{\nu,1}^{(n)} \qquad \left(\text{or } r < \sqrt{j_{\nu,1}^{(n)} j_{\nu,1}^{(n+1)}}\right)$$

for all $\nu > n$, we have

$$\left(j_{\nu,m}^{(n)}\right)\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right) - \left(j_{\nu,m}^{(n+1)}\right)\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right) < 0.$$

Thus, the inequality

$$\frac{d\varphi_{\nu,n}(r)}{dr} = -\left(\frac{1}{\nu - n} - 1\right)\sum_{m\geq 1} \frac{4r\left(j_{\nu,m}^{(n)}\right)^2}{\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right)^2} - \sum_{m\geq 1} \frac{4r\left(j_{\nu,m}^{(n+1)}\right)^2}{\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right)^2}$$

$$< \sum_{m\geq 1} \frac{4r\left(j_{\nu,m}^{(n)}\right)^2}{\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right)^2} - \sum_{m\geq 1} \frac{4r\left(j_{\nu,m}^{(n+1)}\right)^2}{\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right)^2}$$

$$= 4r\sum_{m\geq 1} \frac{\left(j_{\nu,m}^{(n)}\right)^2\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right)^2 - \left(j_{\nu,m}^{(n+1)}\right)^2\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right)^2}{\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right)^2\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right)^2} < 0$$

is satisfied. Consequently, the function $\varphi_{\nu,n}$ is strictly decreasing. We also observe that

$$\lim_{r\searrow 0} \varphi_{\nu,n}(r) = 1 > \beta \qquad \text{and} \qquad \lim_{r\nearrow j_{\nu,1}^{(n)}} \varphi_{\nu,n}(r) = -\infty,$$

which means that, for $z \in \mathbb{D}_{r_1}$, we have

$$\operatorname{Re}\left(1 + \frac{z f''_{\nu,n}(z)}{f'_{\nu,n}(z)}\right) > \beta$$

if and only if r_1 is the unique root of $1 + \frac{r f''_{\nu,n}(r)}{f'_{\nu,n}(r)} = \beta$, located in $(0, j_{\nu,1}^{(n)})$.

(b) Note that

$$1 + \frac{z g''_{\nu,n}(z)}{g'_{\nu,n}(z)} = (n - \nu + 1) + \frac{(n - \nu + 2) z J_\nu^{(n+1)}(z) + z^2 J_\nu^{(n+2)}(z)}{(n - \nu + 1) J_\nu^{(n)}(z) + z J_\nu^{(n+1)}(z)}.$$

By using (1.1) and (2.1), we find

$$g'_{\nu,n}(z) = 2^{\nu}\Gamma(\nu-n+1)z^{n-\nu}\left[(n-\nu+1)J_{\nu}^{(n)}(z) + zJ_{\nu}^{(n+1)}(z)\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m(2m+1)\Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m!\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}\left(\frac{z}{2}\right)^{2m} \tag{3.4}$$

and

$$\lim_{m\to\infty} \frac{m\log m}{\lambda(m,n,\nu)} = \frac{1}{2},$$

where

$$\lambda(m,n,\nu) = [2m\log 2 + \log\Gamma(m+1)$$

$$+ \log\Gamma(2m-n+\nu+1) + \log\Gamma(m+\nu+1)$$

$$- \log\Gamma(2m+\nu+1) - \log\Gamma(\nu-n+1) - \log(2m+1)].$$

Here, we have used the equalities

$$m! = \Gamma(m+1) \qquad \text{and} \qquad \lim_{m\to\infty}\frac{\log\Gamma(am+b)}{m\log m} = a,$$

where a and b are positive constants. Thus, by applying Hadamard's theorem [17, p. 26], we can write the infinite-product representation of $g'_{\nu,n}(z)$ as follows:

$$g'_{\nu,n}(z) = \prod_{m\geq 1}\left(1 - \frac{z^2}{\left(\gamma_{\nu,m}^{(n)}\right)^2}\right), \tag{3.5}$$

where $\gamma_{\nu,m}^{(n)}$ denotes the mth positive zero of the function $g'_{\nu,n}$. According to Lemma 2.5, for $\nu > n-1$, the function $g'_{\nu,n} \in \mathcal{LP}$ and the smallest positive zero of $g'_{\nu,n}$ does not exceed the first positive zero of $J_{\nu}^{(n)}$.

By means of (3.5), we obtain

$$1 + \frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)} = 1 - \sum_{m\geq 1}\frac{2z^2}{\left(\gamma_{\nu,m}^{(n)}\right)^2 - z^2}.$$

If we use inequality (3.2), then, for all $z \in \mathbb{D}_{\gamma_{\nu,m}^{(n)}}$, we get the inequality

$$\operatorname{Re}\left(1 + \frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)}\right) \geq 1 - \sum_{m\geq 1}\frac{2r^2}{\left(\gamma_{\nu,m}^{(n)}\right)^2 - r^2},$$

where $|z| = r$.

Thus, for $r \in (0, \gamma_{\nu,1}^{(n)})$, we get

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)}\right)\right\} = 1+\frac{rg''_{\nu,n}(r)}{g'_{\nu,n}(r)}.$$

A function $G_{\nu,n}\colon (0,\gamma_{\nu,1}^{(n)}) \to \mathbb{R}$ defined by

$$G_{\nu,n}(r) = 1+\frac{rg''_{\nu,n}(r)}{g'_{\nu,n}(r)},$$

is strictly decreasing and, moreover,

$$\lim_{r\searrow 0} G_{\nu,n}(r) = 1 > \beta \qquad \text{and} \qquad \lim_{r\nearrow\gamma_{\nu,1}^{(n)}} G_{\nu,n}(r) = -\infty.$$

Thus, in view of the minimum principle for harmonic functions, for $z \in \mathbb{D}_{r_2}$, we get

$$\operatorname{Re}\left(1+\frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)}\right) > \beta$$

if and only if r_2 is the unique root of $1+\dfrac{rg''_{\nu,n}(r)}{g'_{\nu,n}(r)} = \beta$ located in $(0,\gamma_{\nu,1}^{(n)})$.

(c) Observe that

$$1+\frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)} = \frac{n-\nu+2}{2}+\frac{\sqrt{z}}{2}\frac{(n-\nu+3)J_\nu^{(n+1)}(\sqrt{z})+\sqrt{z}J_\nu^{(n+2)}(\sqrt{z})}{(n-\nu+2)J_\nu^{(n)}(\sqrt{z})+\sqrt{z}J_\nu^{(n+1)}(\sqrt{z})}.$$

By using (1.1) and (2.1), we find

$$\begin{aligned} h'_{\nu,n}(z) &= 2^{\nu-1}\Gamma(\nu-n+1)z^{\frac{n-\nu}{2}}\Big[(n-\nu+2)J_\nu^{(n)}(\sqrt{z})+\sqrt{z}J_\nu^{(n+1)}(\sqrt{z})\Big] \\ &= \sum_{m=0}^{\infty}\frac{(-1)^m(m+1)\Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m!\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}\left(\frac{z}{4}\right)^m \end{aligned} \tag{3.6}$$

and

$$\lim_{m\to\infty}\frac{m\log m}{\tau(m,n,\nu)} = \frac{1}{2},$$

where

$$\begin{aligned} \tau(m,n,\nu) = \big[&2m\log 2+\log\Gamma(m+1) \\ &+\log\Gamma(2m-n+\nu+1)+\log\Gamma(m+\nu+1) \\ &-\log\Gamma(2m+\nu+1)-\log\Gamma(\nu-n+1)-\log(m+1)\big]. \end{aligned}$$

Hence, by applying Hadamard's theorem [17, p. 26], we can write the infinite-product representation of $h'_{\nu,n}(z)$ as follows:

$$h'_{\nu,n}(z) = \prod_{m\geq 1}\left(1 - \frac{z}{\delta^{(n)}_{\nu,m}}\right), \tag{3.7}$$

where $\delta^{(n)}_{\nu,m}$ denotes the mth positive zero of the function $h'_{\nu,n}$. According to Lemma 2.5 for $\nu > n-1$, the function $h'_{\nu,n} \in \mathcal{LP}$ and the smallest positive zero of $h'_{\nu,n}$ does not exceed the first positive zero of $J^{(n)}_{\nu}$.

By virtue of (3.5), we obtain

$$1 + \frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)} = 1 - \sum_{m\geq 1}\frac{z}{\delta^{(n)}_{\nu,m} - z}.$$

By using inequality (3.2), for all $z \in \mathbb{D}_{\delta^{(n)}_{\nu,m}}$, we get the inequality

$$\operatorname{Re}\left(1 + \frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)}\right) \geq 1 - \sum_{m\geq 1}\frac{r}{\delta^{(n)}_{\nu,m} - r},$$

where $|z| = r$. Thus, for $r \in \left(0, \delta^{(n)}_{\nu,1}\right)$, we have

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1 + \frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)}\right)\right\} = 1 + \frac{rh''_{\nu,n}(r)}{h'_{\nu,n}(r)}.$$

The function $H_{\nu,n} \colon \left(0, \delta^{(n)}_{\nu,1}\right) \to \mathbb{R}$ defined by

$$H_{\nu,n}(r) = 1 + \frac{rh''_{\nu,n}(r)}{h'_{\nu,n}(r)}$$

is strictly decreasing and, moreover,

$$\lim_{r\searrow 0} H_{\nu,n}(r) = 1 > \beta \quad \text{and} \quad \lim_{r\nearrow \delta^{(n)}_{\nu,1}} H_{\nu,n}(r) = -\infty.$$

As a result, in view of the minimum principle for harmonic functions with $z \in \mathbb{D}_{r_3}$, we conclude that

$$\operatorname{Re}\left(1 + \frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)}\right) > \beta$$

if and only if r_3 is the unique root of $1 + \dfrac{rh''_{\nu,n}(r)}{h'_{\nu,n}(r)} = \beta$, located in $\left(0, \delta^{(n)}_{\nu,1}\right)$.

Theorem 3.2 is proved.

In view of Theorem 3.2, we tabulate the radii of convexity for $f_{\nu,n},\ g_{\nu,n}$ and $h_{\nu,n}$ for fixed $\nu = 3.5,\ n = 0, 1, 2, 3,$ and, respectively, $\beta = 0$ and $\beta = 0.5$. These values are presented in Table 3.2. Moreover, in Table 3.2,

Table 3.2

n	$r^c_\beta(f_{3.5,n})$		$r^c_\beta(g_{3.5,n})$		$r^c_\beta(h_{3.5,n})$	
	$\beta=0$	$\beta=0.5$	$\beta=0$	$\beta=0.5$	$\beta=0$	$\beta=0.5$
0	2.7183	2.0865	0.5234	1.1461	6.2189	3.7194
1	1.8179	1.3998	1.2017	0.9084	3.7394	2.2873
2	1.0592	0.8123	0.8833	0.6715	1.9450	1.2190
3	0.4141	0.3131	0.5683	0.4350	0.7726	0.4968

we see that the radius of convexity is decreasing as a function of the order of the derivatives and the order of convexity. In other words, all these results imply that

$$r^c_\beta(f_{\nu,0}) > r^c_\beta(f_{\nu,1}) > r^c_\beta(f_{\nu,2}) > \ldots > r^c_\beta(f_{\nu,n}) > \ldots$$

for $\beta \in [0,1)$ and $\nu > n-1,\ n \in \mathbb{N}_0$. In addition, we can write

$$r^c_{\beta_1}(f_{\nu,n}) < r^c_{\beta_0}(f_{\nu,n})$$

for $0 \le \beta_0 < \beta_1 < 1$ and $\nu > n-1,\ n \in \mathbb{N}_0$. The same inequalities are also true for $r^c_\beta(g_{\nu,n})$ and $r^c_\beta(h_{\nu,n})$.

For $n = 0$ in Theorem 3.2, we get the results obtained by Baricz and Szász [4]. Our results is a common generalization of these results.

3.2. Bounds for the Radii of Starlikeness and Convexity of the Functions $g_{\nu,n}$ and $h_{\nu,n}$. In this section, we consider two different functions $g_{\nu,n}$ and $h_{\nu,n}$ that are normalized forms of the derivatives of the Bessel function of the first kind given by (1.1). In this case, our first aim is to show that the radii of univalence of these functions correspond to the radii of starlikeness.

Theorem 3.3. *The following inequalities hold:*

(a) If $\nu > n-1$, then $r^*(g_{\nu,n})$ satisfies the inequalities

$$r^*(g_{\nu,n}) < \sqrt{2}\sqrt{a^{-1}_{\nu,n}},$$

$$\frac{2\sqrt{3}}{3}\sqrt{a^{-1}_{\nu,n}} < r^*(g_{\nu,n}) < 2\sqrt{3}\sqrt{\frac{1}{9a_{\nu,n}-5b_{\nu,n}}}.$$

(b) If $\nu > n-1$, then $r^*(h_{\nu,n})$ satisfies the inequalities

$$r^*(h_{\nu,n}) < 2a^{-1}_{\nu,n},$$

$$2a_{\nu,n}^{-1} < r^*(h_{\nu,n}) < \frac{8}{4a_{\nu,n} - 3b_{\nu,n}},$$

where

$$a_{\nu,n} = \frac{\nu+2}{(\nu-n+2)(\nu-n+1)}$$

and

$$b_{\nu,n} = \frac{(\nu+4)(\nu+3)}{(\nu-n+4)(\nu-n+3)(\nu+2)}.$$

Proof. (a) By using the first Rayleigh sum (2.6) and the implicit relation for $r^*(g_{\nu,n})$, obtained by Kreyszing and Todd [16], for all $\nu > n-1$, we get

$$\frac{1}{(r^*(g_{\nu,n}))^2} = \sum_{m\geq 1} \frac{2}{\left(j_{\nu,m}^{(n)}\right)^2 - (r^*(g_{\nu,n}))^2}$$

$$> \sum_{m\geq 1} \frac{2}{\left(j_{\nu,m}^{(n)}\right)^2} = \frac{\nu+2}{2(\nu-n+2)(\nu-n+1)}.$$

Further, by using the Euler–Rayleigh inequalities, it is possible to get tighter bounds for the radius of univalence (and starlikeness) $r^*(g_{\nu,n})$. We define a function $\Psi_{\nu,n}(z) = g'_{\nu,n}(z)$, where $g'_{\nu,n}$ is defined by (3.5). Thus, by taking the logarithmic derivative of both sides of (3.5) for $|z| < \gamma_{\nu,1}^{(n)}$, we obtain

$$\begin{aligned}\frac{\Psi'_{\nu,n}(z)}{\Psi_{\nu,n}(z)} &= -\sum_{m\geq 1} \frac{2z}{\left(\gamma_{\nu,m}^{(n)}\right)^2 - z^2} \\ &= -2\sum_{m\geq 1}\sum_{k\geq 0} \frac{1}{\left(\gamma_{\nu,m}^{(n)}\right)^{2(k+1)}} z^{2k+1} \\ &= -2\sum_{k\geq 0} \sigma_{k+1} z^{2k+1}, \end{aligned} \tag{3.8}$$

where

$$\sigma_k = \sum_{m\geq 1} \left(\gamma_{\nu,m}^{(n)}\right)^{-k}$$

is the Euler–Rayleigh sum for the zeros of $\Psi_{\nu,n}$. In addition, by using (3.4), from the infinite-sum representation of $\Psi_{\nu,n}$, we obtain

$$\frac{\Psi'_{\nu,n}(z)}{\Psi_{\nu,n}(z)} = \frac{\sum_{m\geq 0} U_m z^{2m+1}}{\sum_{m\geq 0} V_m z^{2m}}, \tag{3.9}$$

where

$$U_m = \frac{2(-1)^{m+1}\Gamma(2m+\nu+3)\Gamma(\nu-n+1)(2m+3)}{m!4^{m+1}\Gamma(2m-n+\nu+3)\Gamma(m+\nu+2)}$$

and

$$V_m = \frac{(-1)^{m}\Gamma(2m+\nu+1)\Gamma(\nu-n+1)(2m+1)}{m!4^{m}\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}.$$

By comparing the coefficients with the same degrees of (3.8) and (3.9), we arrive at the Euler–Rayleigh sums

$$\sigma_1 = \frac{3(\nu+2)}{4(\nu-n+2)(\nu-n+1)}$$

and

$$\sigma_2 = \frac{3(\nu+2)}{16(\nu-n+2)(\nu-n+1)} \times \left(\frac{3(\nu+2)}{(\nu-n+2)(\nu-n+1)} - \frac{5(\nu+4)(\nu+3)}{3(\nu-n+4)(\nu-n+3)(\nu+2)}\right).$$

By virtue of the Euler–Rayleigh inequalities

$$\sigma_k^{-\frac{1}{k}} < \left(\gamma_{\nu,1}^{(n)}\right)^2 < \frac{\sigma_k}{\sigma_{k+1}}$$

for $\nu > n-1,\ k \in \mathbb{N}$ and $k = 1,$ we arrive at the following inequality:

$$\frac{4(\nu-n+2)(\nu-n+1)}{3(\nu+2)} < \left(r^*(g_{\nu,n})\right)^2 < \frac{4}{\frac{3(\nu+2)}{(\nu-n+2)(\nu-n+1)} - \frac{5(\nu+4)(\nu+3)}{3(\nu-n+4)(\nu-n+3)(\nu+2)}},$$

and it is possible to get tighter bounds for the other values of $k \in \mathbb{N}$.

(b) By using the first Rayleigh sum (2.6) and the implicit relation for $r^*(h_{\nu,n})$ obtained by Kreyszing and Todd [16], for all $\nu > n-1,$ we get

$$\frac{1}{r^*(h_{\nu,n})} = \sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^2 - r^*(h_{\nu,n})} > \sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^2} = \frac{\nu+2}{2(\nu-n+2)(\nu-n+1)}.$$

Further, by using the Euler–Rayleigh inequalities, it is possible to get tighter bounds for the radius of univalence (and starlikeness) $r^*(h_{\nu,n})$. We define a function

$$\Phi_{\nu,n}(z) = h'_{\nu,n}(z),$$

where $h'_{\nu,n}$ is given by (3.6) or (3.7). Thus, taking the logarithmic derivatives of both sides of (3.7), we obtain

$$\begin{aligned}\frac{\Phi'_{\nu,n}(z)}{\Phi_{\nu,n}(z)} &= -\sum_{m\geq 1}\frac{1}{\delta^{(n)}_{\nu,m}-z}\\ &= -\sum_{m\geq 1}\sum_{k\geq 0}\frac{1}{\left(\delta^{(n)}_{\nu,m}\right)^{k+1}}z^k\\ &= -\sum_{k\geq 0}\rho_{k+1}z^k, \quad |z|<\delta^{(n)}_{\nu,1},\end{aligned} \tag{3.10}$$

where

$$\rho_k = \sum_{m\geq 1}\left(\delta^{(n)}_{\nu,m}\right)^{-k}$$

is the Euler–Rayleigh sum for the zeros of $\Phi_{\nu,n}$. Moreover, by using (3.6), from the infinite-sum representation of $\Phi_{\nu,n}$, we obtain

$$\frac{\Phi'_{\nu,n}(z)}{\Phi_{\nu,n}(z)} = \frac{\sum_{m\geq 0}K_m z^m}{\sum_{m\geq 0}L_m z^m}, \tag{3.11}$$

where

$$K_m = \frac{(-1)^{m+1}\Gamma(2m+\nu+3)\Gamma(\nu-n+1)(m+2)}{m!4^{m+1}\Gamma(2m-n+\nu+3)\Gamma(m+\nu+2)}$$

and

$$L_m = \frac{(-1)^{m}\Gamma(2m+\nu+1)\Gamma(\nu-n+1)(m+1)}{m!4^{m}\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}.$$

By comparing the coefficients with the same degrees of (3.10) and (3.11), we get the following Euler–Rayleigh sums:

$$\rho_1 = \frac{\nu+2}{2(\nu-n+2)(\nu-n+1)}$$

and

$$\begin{aligned}\rho_2 = {}&\frac{\nu+2}{4(\nu-n+2)(\nu-n+1)}\\ &\times\left(\frac{\nu+2}{(\nu-n+2)(\nu-n+1)}-\frac{3(\nu+4)(\nu+3)}{4(\nu-n+4)(\nu-n+3)(\nu+2)}\right).\end{aligned}$$

If we use the Euler–Rayleigh inequalities

$$\rho_k^{-\frac{1}{k}} < \delta_{\nu,1}^{(n)} < \frac{\rho_k}{\rho_{k+1}}$$

for $\nu > n-1,\ k \in \mathbb{N}$ and $k=1,$ then we obtain the following inequality:

$$\frac{2(\nu-n+2)(\nu-n+1)}{\nu+2} < r^*(h_{\nu,n}) < \frac{2}{\dfrac{\nu+2}{(\nu-n+2)(\nu-n+1)} - \dfrac{3(\nu+4)(\nu+3)}{4(\nu-n+4)(\nu-n+3)(\nu+2)}},$$

and it is possible to get tighter bounds for other values of $k \in \mathbb{N}$.

Theorem 3.3 is proved.

If we take $n=0$ in Theorem 3.3, then we get the results obtained by Aktaş, et al. [1]. Our results is a common generalization of the indicated results. For special cases of the parameters ν and $n,$ Theorem 3.3 reduces tight lower and upper bounds for the radii of starlikeness and convexity of numerous elementary functions. Thus, for $\nu = \frac{3}{2}$ and $n=2$ in Theorem 3.3, we have

$$\sqrt{\frac{2}{7}} < r^*\Big(g_{\frac{3}{2},2}(z) = 4\sin z - 4z\cos z\Big) < \sqrt{\frac{3}{7}}$$

and

$$\frac{3}{7} < r^*\Big(h_{\frac{3}{2},2}(z) = 4\sqrt{z}\sin\sqrt{z} - 4z\cos\sqrt{z}\Big) < \frac{2940}{5969}.$$

The next result establishes the bounds for the radii of convexity of the functions $g_{\nu,n}$ and $h_{\nu,n}.$

Theorem 3.4. *The following statements hold:*

(a) If $\nu > n-1,$ then $r^c(g_{\nu,n})$ satisfies the inequalities

$$\frac{2}{3}\sqrt{a_{\nu,n}^{-1}} < r^c(g_{\nu,n}) < 6\sqrt{\frac{1}{81a_{\nu,n} - 25b_{\nu,n}}}.$$

(b) If $\nu > n-1,$ then $r^c(h_{\nu,n})$ satisfies the inequalities

$$a_{\nu,n}^{-1} < r^c(h_{\nu,n}) < \frac{16}{16a_{\nu,n} - 9b_{\nu,n}},$$

where $a_{\nu,n}$ and $b_{\nu,n}$ are given in Theorem 3.3.

Proof. (a) By using the Alexander duality theorem for starlike and convex functions, we can say that the function $g_{\nu,n}(z)$ is convex if and only if $zg'_{\nu,n}(z)$ is starlike. However, the smallest positive zero of $z \mapsto z(zg'_{\nu,n}(z))'$

is actually the radius of starlikeness of $z \mapsto (zg'_{\nu,n}(z))$ according to Theorems 3.1 and 3.2. Therefore, the radius of convexity $r^c(g_{\nu,n})$ is the smallest positive root of the equation $(zg'_{\nu,n}(z))' = 0$. Hence, it follows from (3.4) that

$$\Delta_{\nu,n}(z) = (zg'_{\nu,n}(z))' = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+1)^2\Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m!4^m\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} z^{2m}.$$

Since the function $g_{\nu,n}(z)$ belongs to the Laguerre–Pólya class of entire functions and $\mathcal{LP}$ is closed under differentiation, we can say that the function $\Delta_{\nu,n}(z) \in \mathcal{LP}$. Therefore, all zeros of the function $\Delta_{\nu,n}$ are real. Suppose that $d^{(n)}_{\nu,m}$ are zeros of the function $\Delta_{\nu,n}$. Then the function $\Delta_{\nu,n}$ has the following infinite-product representation:

$$\Delta_{\nu,n}(z) = \prod_{m\geq 1} \left(1 - \frac{z^2}{\left(d^{(n)}_{\nu,m}\right)^2}\right). \tag{3.12}$$

By taking the logarithmic derivative of (3.12), we get

$$\begin{aligned}\frac{\Delta'_{\nu,n}(z)}{\Delta_{\nu,n}(z)} &= -2\sum_{m\geq 1} \frac{z}{\left(d^{(n)}_{\nu,m}\right)^2 - z^2} \\ &= -2\sum_{m\geq 1}\sum_{k\geq 0} \frac{1}{\left(d^{(n)}_{\nu,m}\right)^{2(k+1)}} z^{2k+1} \\ &= -2\sum_{k\geq 0} \kappa_{k+1} z^{2k+1}, \quad |z| < d^{(n)}_{\nu,1}, \end{aligned} \tag{3.13}$$

where

$$\kappa_k = \sum_{m\geq 1} \left(d^{(n)}_{\nu,m}\right)^{-k}$$

is the Euler–Rayleigh sum for the zeros of $\Delta_{\nu,n}$. On the other hand, by considering the infinite-sum representation of $\Delta_{\nu,n}(z)$, we obtain

$$\frac{\Delta'_{\nu,n}(z)}{\Delta_{\nu,n}(z)} = \frac{\sum_{m\geq 0} X_m z^{2m+1}}{\sum_{m\geq 0} Y_m z^{2m}}, \tag{3.14}$$

where

$$X_m = \frac{2(-1)^{m+1}\Gamma(2m+\nu+3)\Gamma(\nu-n+1)(2m+3)^2}{m!4^{m+1}\Gamma(2m-n+\nu+3)\Gamma(m+\nu+2)}$$

and

$$Y_m = \frac{(-1)^m\Gamma(2m+\nu+1)\Gamma(\nu-n+1)(2m+1)^2}{m!4^m\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}.$$

By comparing the coefficients of (3.13) and (3.14), we find

$$\kappa_1 = \frac{9(\nu+2)}{4(\nu-n+2)(\nu-n+1)}$$

and

$$\kappa_2 = \frac{9(\nu+2)}{16(\nu-n+2)(\nu-n+1)} \times \left(\frac{9(\nu+2)}{(\nu-n+2)(\nu-n+1)} - \frac{25(\nu+4)(\nu+3)}{9(\nu-n+4)(\nu-n+3)(\nu+2)} \right).$$

By using the Euler–Rayleigh inequalities

$$\kappa_k^{-\frac{1}{k}} < \left(d_{\nu,1}^{(n)}\right)^2 < \frac{\kappa_k}{\kappa_{k+1}}$$

for $\nu > n-1,\ k \in \mathbb{N}$, and $k=1,$ we get the inequality

$$\frac{4(\nu-n+2)(\nu-n+1)}{9(\nu+2)} < \left(r^c(g_{\nu,n})\right)^2 < \frac{4}{\dfrac{9(\nu+2)}{(\nu-n+2)(\nu-n+1)} - \dfrac{25(\nu+4)(\nu+3)}{9(\nu-n+4)(\nu-n+3)(\nu+2)}}.$$

Moreover, it is possible to get tighter bounds for other values of $k \in \mathbb{N}$.

(b) By using the same procedure as in the previous proof, we can show that the radius of convexity $r^c(h_{\nu,n})$ is the smallest positive root of the equation

$$\left(zh'_{\nu,n}(z)\right)' = 0$$

according to Theorem 3.2. From (3.6), we get

$$\Theta_{\nu,n}(z) = \left(zh'_{\nu,n}(z)\right)' = \sum_{m=0}^{\infty} \frac{(-1)^m (m+1)^2 \Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m! 4^m \Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} z^m. \tag{3.15}$$

Moreover, we know that $h_{\nu,n}(z)$ belongs to the Laguerre–Pólya class of entire functions $\mathcal{LP}$ and, consequently, $\Theta_{\nu,n}(z) \in \mathcal{LP}$. In other words, all zeros of the function $\Theta_{\nu,n}$ are real. Assume that $l_{\nu,m}^{(n)}$ are the zeros of the function $\Theta_{\nu,n}$. In this case, the function $\Theta_{\nu,n}$ has the following infinite-product representation:

$$\Theta_{\nu,n}(z) = \prod_{m\geq 1} \left(1 - \frac{z^2}{\left(l_{\nu,m}^{(n)}\right)^2}\right). \tag{3.16}$$

By taking the logarithmic derivatives of both sides of (3.16) for $|z| < l_{\nu,1}^{(n)}$, we get

$$\begin{aligned}\frac{\Theta'_{\nu,n}(z)}{\Theta_{\nu,n}(z)} &= -\sum_{m\geq 1}\frac{1}{l_{\nu,m}^{(n)} - z}\\ &= -\sum_{m\geq 1}\sum_{k\geq 0}\frac{1}{\left(l_{\nu,m}^{(n)}\right)^{k+1}}z^k\\ &= -\sum_{k\geq 0}\omega_{k+1}z^k, \end{aligned}\tag{3.17}$$

where

$$\omega_k = \sum_{m\geq 1}\left(l_{\nu,m}^{(n)}\right)^{-k}.$$

In addition, by using the derivative of the infinite-sum representation for the infinite-sum representation of (3.15), we obtain

$$\frac{\Theta'_{\nu,n}(z)}{\Theta_{\nu,n}(z)} = \sum_{m\geq 0}T_m z^m \Big/ \sum_{m\geq 0}S_m z^m, \tag{3.18}$$

where

$$T_m = \frac{(-1)^{m+1}\Gamma(2m+\nu+3)\Gamma(\nu-n+1)(m+2)^2}{m!4^{m+1}\Gamma(2m-n+\nu+3)\Gamma(m+\nu+2)}$$

and

$$S_m = \frac{(-1)^m\Gamma(2m+\nu+1)\Gamma(\nu-n+1)(m+1)^2}{m!4^m\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}.$$

By comparing the coefficients of (3.17) and (3.18), we get

$$\omega_1 = \frac{\nu+2}{(\nu-n+2)(\nu-n+1)}$$

and

$$\begin{aligned}\omega_2 = {}&\frac{\nu+2}{(\nu-n+2)(\nu-n+1)}\\ &\times\left(\frac{\nu+2}{(\nu-n+2)(\nu-n+1)} - \frac{9(\nu+4)(\nu+3)}{16(\nu-n+4)(\nu-n+3)(\nu+2)}\right).\end{aligned}$$

By using the Euler–Rayleigh inequalities

$$\omega_k^{-\frac{1}{k}} < l_{\nu,1}^{(n)} < \frac{\omega_k}{\omega_{k+1}}$$

for $\nu > n-1,\ k\in\mathbb{N}$, and $k=1$, we arrive at the following inequality:

$$\frac{(\nu-n+2)(\nu-n+1)}{\nu+2} < r^c(h_{\nu,n})$$

$$< \frac{1}{\frac{\nu+2}{(\nu-n+2)(\nu-n+1)} - \frac{9(\nu+4)(\nu+3)}{16(\nu-n+4)(\nu-n+3)(\nu+2)}}.$$

Moreover, it is possible to get tighter bounds for the other values of $k \in \mathbb{N}$.

Theorem 3.4 is proved.

If we take $n = 0$ in Theorem 3.4, then we get the results obtained by Aktaş, et al. [2]. For special cases $n = 1, 2, 3,$ we arrive at the following result:

Corollary 3.1. *The following inequalities hold:*

$$\frac{2}{3}\sqrt{a_{\nu,1}^{-1}} < r^c(g_{\nu,1}) < 6\sqrt{\frac{1}{81a_{\nu,1} - 25b_{\nu,1}}}, \quad \nu > 0,$$

$$a_{\nu,1}^{-1} < r^c(h_{\nu,1}) < \frac{16}{16a_{\nu,1} - 9b_{\nu,1}}, \quad \nu > 0,$$

$$\frac{2}{3}\sqrt{a_{\nu,2}^{-1}} < r^c(g_{\nu,2}) < 6\sqrt{\frac{1}{81a_{\nu,2} - 25b_{\nu,2}}}, \quad \nu > 1,$$

$$a_{\nu,2}^{-1} < r^c(h_{\nu,2}) < \frac{16}{16a_{\nu,2} - 9b_{\nu,2}}, \quad \nu > 1,$$

$$\frac{2}{3}\sqrt{a_{\nu,3}^{-1}} < r^c(g_{\nu,3}) < 6\sqrt{\frac{1}{81a_{\nu,3} - 25b_{\nu,3}}}, \quad \nu > 2,$$

$$a_{\nu,3}^{-1} < r^c(h_{\nu,3}) < \frac{16}{16a_{\nu,3} - 9b_{\nu,3}}, \quad \nu > 2,$$

where $a_{\nu,n}$ and $b_{\nu,n}$ for $n = 1, 2, 3$ are given in Theorem 3.3.

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