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On the Janowski class of generalized Struve functions

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Abstract
In this paper, we are mainly interested to find the sufficient conditions on parameters A, B, b and c that will ensure the generalized Struve function $u_{v,b,c}$ satisfies the subordination $u_{v,b,c}(z) \prec (1 + Az)/(1 + Bz)$.

Keywords Struve functions · Differential subordination · Janowski functions

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1 Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. An analytic function f is subordinate to an analytic function g (written as $f \prec g$) if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$ such that $f(z) = g(w(z))$. In particular if g is univalent in $\mathcal{U}$, then $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. Let $\mathcal{P}[A, B]$ denote the class of analytic functions p such that $p(0) = 1$ and

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \quad -1 \le B < A \le 1, \ z \in \mathcal{U}.$$

Note that for $0 \le \beta < 1$, $\mathcal{P}[1 - 2\beta, -1]$ is the class of analytic functions p with $p(0) = 1$ satisfying $\operatorname{Re} p(z) > \beta$ in $\mathcal{U}$.

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For $-1 \leq B < A \leq 1$, the class $\mathcal{S}^*[A, B]$ defined by

$$\mathcal{S}^*[A, B] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz},\ z \in \mathcal{U} \right\}$$

is the class of Janowski starlike functions [6]. For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the usual class of starlike functions of order β;

$$\mathcal{S}^*[1 - \beta, 0] := \mathcal{S}^*_\beta = \{ f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta \}$$

and

$$\mathcal{S}^*[\beta, -\beta] := \mathcal{S}^*[\beta] = \{ f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta |zf'(z)/f(z) + 1| \}.$$

These classes has been studied in [2,3]. A function $f \in \mathcal{A}$ is said to be close-to-convex of order β with respect to a function $g \in \mathcal{S}^*$ if $\mathrm{Re} \frac{zf'(z)}{g(z)} > \beta$.

The struve functions H_v appeared as special solutions of the second order inhomogeneous differential equations of the form

$$z^2 w''(z) + zw(z) + (z^2 - v^2)w(z) = \frac{4 \left(\frac{1}{2}z\right)^{v+1}}{\sqrt{\pi}\Gamma\left(v + \frac{1}{2}\right)} \tag{1.1}$$

known as inhomogeneous Bessel differential equation. The solutions are given by

$$w(z) = H_v(z) + c_1 J_v(z) + c_2 Y_v(z),$$

where c_1, c_2 are arbitrary constants, J_v and Y_v are Bessel functions of the first and second kinds, H_v is Struve functions. In the solution of Eq. (1.1), a function appeared in an article by H. Struve (1882), was later ascribed Struve's name and the special notation H_v. It is defined as

$$H_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+v+1}}{\Gamma\left(n + \frac{3}{2}\right)\Gamma\left(n + v + \frac{3}{2}\right)},$$

where $\Gamma(z)$ is the gamma function. However, the modified Struve function L_v appeared into mathematical literature by Nicholson [11, p. 218]. Modified Struve function L_v special solutions of the second order inhomogeneous differential equations of the form

$$z^2 w''(z) + zw(z) - (z^2 + v^2)w(z) = \frac{4 \left(\frac{1}{2}z\right)^{v+1}}{\sqrt{\pi}\Gamma\left(v + \frac{1}{2}\right)} \tag{1.2}$$

and its power series representation is given by

$$L_v(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+v+1}}{\Gamma\left(n + \frac{3}{2}\right)\Gamma\left(n + v + \frac{3}{2}\right)}.$$

Applications of Struve functions occur in water-wave and surface-wave problems, unsteady aerodynamics, resistive MHD instability theory and optical diffraction. More recently Struve functions have appeared in many particle quantum dynamical studies of spin decoherence and nanotubes. For some details see [1,12].

Now consider the second order inhomogenous differential equation

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - v^2 + (1 - b)v\right] w(z) = \frac{4 \left(\frac{z}{2}\right)^{v+1}}{\sqrt{\pi}\Gamma\left(v + \frac{b}{2}\right)}, \tag{1.3}$$

where $b,\ c,\ v\ \in \mathbb{C}$. Equation (1.3) generalizes the Eqs. (1.1) and (1.2). In particular for $b=c=1$ and $b=c+2=1$ we obtain (1.1) and (1.2), respectively. Its particular solution has the series form

$$w_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n c^n \left(\frac{1}{2}z\right)^{2n+v+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+v+\frac{b+2}{2}\right)}. \tag{1.4}$$

It is known as Generalized Struve function of order v. Consider the transformation

$$\begin{aligned} u_{v,b,c}(z) &= 2^v\sqrt{\pi}\Gamma(v+(b+2)/2)z^{(-v-1)/2}w_{v,b,c}(\sqrt{z}) \\ &= \sum_{n=0}^{\infty}\frac{(-c/4)^n z^n}{(3/2)_n\,(k)_n} \end{aligned} \tag{1.5}$$

where $b,\ c,\ v\in\mathbb{C}$ and $k=v+(b+2)/2\neq 0,-1,-2,-3,\ldots$ with

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1, & n=0,\ \gamma\in\mathbb{C}\backslash\{0\},\\ \gamma(\gamma+1)\ldots(\gamma+n-1), & n\in\mathbb{N},\ \gamma\in\mathbb{C}.\end{cases}$$

The function $u_{v,b,c}$ is analytic in $\mathcal{U}$ and is the solution of the differential equation

$$4z^2u''(z)+2(2k+1)zu'(z)+(cz+2k-2)u(z)=2k-2 \tag{1.6}$$

and satisfies the relation

$$u_v(z)+2zu_v'(z)+\frac{cz}{2k}u_{v+1}(z)=1.$$

The function $u_v(z)=u_{v,b,c}$ is introduced and studied by Orhan and Yagmur [13]. There has been several works [5,13,15,18,19], studying geometric properties of the Struve function, such as on its close-to-convexity, starlikeness, and convexity, radius of starlikeness and convexity.

Recently Ali et al. [4], Mondal and Dhuain [9], and Radhika et al. [14] studied the close-to-convexity, starlikeness of generalized Bessel function in Janowski class. Sangal and Swaminathan [16] studied the starlikeness of Gaussian Hypergeometric functions. Mondal and Dhuain [10] discussed the Janowski class of generalized Bessel–Struve Kernal function. For the sufficient conditions about the Janowski function also see [2,3,17].

This paper studies the Struve function $u_v(z)$ given by the power series (1.5). Sufficient conditions on the parameters A, B, c, b are determined that ensure the Struve function $u_v(z)$ to satisfy the subordination relation $u_v(z)\prec\frac{1+Az}{1+Bz}$. The advantage of such generalized assertions is that, even by handed selections of the parameters A and B, they direct towards many previously renowned applications of this special function. Sufficient conditions are also obtained for $\frac{2k}{cz}\left[1-2u_v'(z)-u_v(z)\right]\in\mathcal{P}[A,B]$.

The following lemma is needed for our main results.

Lemma 1.1 [7,8] *Let $\Omega\subset\mathbb{C}$, and $\Psi:\mathbb{C}^2\times\mathcal{U}\to\mathbb{C}$ satisfy*

$$\Psi(i\rho,\sigma;z)\notin\Omega$$

whenever $z\in\mathcal{U}$, ρ real, $\sigma\leq-(1+\rho^2)/2$. If p is analytic in $\mathcal{U}$ with $p(0)=1$, and $\Psi(p(z),zp'(z);z)\in\Omega$ for $z\in\mathcal{U}$, then Re $p(z)>0$ *in $\mathcal{U}$. In the case $\Psi:\mathbb{C}^3\times\mathcal{U}\to\mathbb{C}$, then the condition in Lemma 1.1 generalized to*

$$\Psi(i\rho,\sigma,\ \mu+i\nu;z)\notin\Omega$$

ρ real, $\sigma+\mu\leq0$ and $\sigma\leq-(1+\rho^2)/2$.

2 Inclusion of generalized Struve function in the Janowski class

Theorem 2.1 *Let* $-1 \leq B \leq 3-2\sqrt{2} \approx 0.171573$. *Suppose* $B < A \leq 1$, *and* $b, c \in \mathbb{R}$ *satisfy*

$$\left(k-\frac{1}{2}\right)+\frac{(k-1)(1+B)}{2} \geq \max\left\{0, \frac{(1+A)(1+B)}{4(A-B)}|c|\right\}. \tag{2.1}$$

Further let A, B, b *and* c *satisfy either the inequality*

$$\begin{aligned}&\left[\left(k-\tfrac{1}{2}\right)^2+\left(k-\tfrac{1}{2}\right)\tfrac{(1+B)}{(1-B)}+\tfrac{(1+B)^2}{(1-B)}\tfrac{(k-1)}{2}+\left(k-\tfrac{1}{2}\right)(k-1)-\tfrac{(k-1)^2(B^2-1)}{4}\right]\\&-\left|\left[\left(k-\tfrac{1}{2}\right)\tfrac{(A+B)}{2(A-B)}+\tfrac{(1+A)(1+B)^2}{4(A-B)(1-B)}+\tfrac{A(1-B^2)}{4(A-B)}(k-1)\right]c\right| \geq \tfrac{(1-A^2)(1-B^2)}{16(A-B)^2}c^2,\end{aligned} \tag{2.2}$$

whenever

$$\begin{aligned}&\left|\left[2\left(k-\frac{1}{2}\right)(A+B)(1-B)+(1+A)(1+B)^2+A(1-B)(1-B^2)(k-1)\right]c\right|\\&\geq \frac{1}{2}(A-B)(1-B)c^2,\end{aligned} \tag{2.3}$$

or the inequality

$$\begin{aligned}&\left[\left(k-\frac{1}{2}\right)\frac{(A+B)}{2(A-B)}+\frac{(1+A)(1+B)^2}{4(A-B)(1-B)}+\frac{A(1-B^2)}{4(A-B)}(k-1)\right]^2 c^2\\&\leq \frac{c^2}{4}\left[\begin{array}{c}\left(k-\tfrac{1}{2}\right)^2+\left(k-\tfrac{1}{2}\right)\tfrac{(1+B)}{(1-B)}+\tfrac{(1+B)^2}{(1-B)}\tfrac{(k-1)}{2}\\+\left(k-\tfrac{1}{2}\right)(k-1)-\tfrac{(k-1)^2(B^2-1)}{4}-\tfrac{(1-AB)^2}{16(A-B)^2}c^2\end{array}\right],\end{aligned} \tag{2.4}$$

whenever

$$\begin{aligned}&\left|\left[2\left(k-\frac{1}{2}\right)(A+B)(1-B)+(1+A)(1+B)^2+A(1-B)(1-B^2)(k-1)\right]c\right|\\&< \frac{1}{2}(A-B)(1-B)c^2.\end{aligned} \tag{2.5}$$

If $(1+B)\,u_v \neq (1+A)$, *then* $u_v(z) \in \mathcal{P}[A, B]$.

Proof Define the analytic function $p : \mathcal{U} \to \mathbb{C}$ by

$$p(z) = -\frac{(1-A)-(1-B)\,u_v(z)}{(1+A)-(1+B)\,u_v(z)}.$$

Then simple computation yields

$$u_v(z) = \frac{(1-A)+(1+A)\,p(z)}{(1-B)+(1+B)\,p(z)}, \tag{2.6}$$

therefore

$$u_v'(z) = \frac{2(A-B)\,p'(z)}{((1-B)+(1+B)\,p(z))^2}, \tag{2.7}$$

and

$$u_v''(z) = \frac{2(A-B)((1-B)+(1+B)\,p(z))\,p''(z)-4(1+B)(A-B)\,p'^2(z)}{((1-B)+(1+B)\,p(z))^3}. \tag{2.8}$$

Thus, using the Eqs. (2.6)–(2.8), the differential equation (1.6) can be rewritten as

$$\begin{aligned} & z^2 p''(z) - \frac{2z^2(1+B)p'^2(z)}{(1-B)+(1+B)p(z)} + \frac{z(2k+1)p'(z)}{2} \\ & \quad + \left[\frac{(1+A)(1+B)cz}{8(A-B)} + \frac{(k-1)(1+B)}{4}\right] p^2(z) \\ & \quad + \left[\frac{(1-AB)cz}{4(A-B)} - \frac{B(k-1)}{2}\right] p(z) + \left[\frac{(1-A)(1-B)cz}{8(A-B)} + \frac{(k-1)(B-1)}{4}\right] \\ & = 0. \end{aligned} \tag{2.9}$$

Assume $\Omega = \{0\}$, and define $\Psi(r, s, t; z)$ by

$$\begin{aligned} \Psi(r, s, t; z) = t & - \frac{2s^2(1+B)}{(1-B)+(1+B)r} + \frac{(2k+1)}{2} s \\ & + r^2 \left[\frac{(1+A)(1+B)}{8(A-B)} cz + \frac{(k-1)(1+B)}{4}\right] \\ & + r \left[\frac{(1-AB)cz}{4(A-B)} - \frac{B(k-1)}{2}\right] \\ & + \left[\frac{(1-A)(1-B)cz}{8(A-B)} + \frac{(k-1)(B-1)}{4}\right]. \end{aligned} \tag{2.10}$$

It follows from (2.9) that $\Psi(p(z),\ zp'(z),\ z^2p''(z);\ z) \in \Omega$. To ensure $\operatorname{Re} p(z) > 0$ for $z \in \mathcal{U}$, from Lemma 1.1, it is enough to establish $\operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) < 0$ in $\mathcal{U}$ for any real ρ, $\sigma \leq -(1+\rho^2)/2$, and $\sigma + \mu \leq 0$. With $z = x + iy \in \mathcal{U}$ in (2.10), a computation yields

$$\begin{aligned} & \operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) \\ & = \mu - \frac{2(1+B)(1-B)\sigma^2}{(1-B)^2 + (1+B)^2\rho^2} + \frac{1}{2}(2k+1)\sigma \\ & \quad - \left[\frac{(1+A)(1+B)}{8(A-B)} cx + \frac{(k-1)(1+B)}{4}\right] \rho^2 \\ & \quad - cy\frac{(1-AB)}{4(A-B)}\rho + \left[\frac{(1-A)(1-B)cx}{8(A-B)} + \frac{(k-1)(B-1)}{4}\right]. \end{aligned} \tag{2.11}$$

Since $\sigma \leq -(1+\rho^2)/2$, and $B \in [-1, 3-2\sqrt{2}]$,

$$\frac{2\left(1-B^2\right)}{(1-B)^2 + (1+B)^2\rho^2}\sigma^2 \geq \frac{2\left(1-B^2\right)}{(1-B)^2 + (1+B)^2\rho^2}\left(\frac{1+\rho^2}{2}\right)^2 \geq \frac{1+B}{2(1-B)}.$$

Thus

$$\begin{aligned}
&\mathrm{Re}\,\Psi(i\rho,\sigma,\mu+i\nu;z)\\
&\le\left\{\mu+\frac{1}{2}(2k+1)\sigma\right\}-\frac{(1+B)}{2(1-B)}-\left[\frac{(1+A)(1+B)}{8\,(A-B)}cx+\frac{(k-1)(1+B)}{4}\right]\rho^2\\
&\quad-\frac{(1-AB)\rho}{4(A-B)}yc+\left[\frac{(1-A)(1-B)cx}{8(A-B)}+\frac{(k-1)\,(B-1)}{4}\right]\\
&\le\{\mu+\sigma\}-\frac{1}{4}\,(2k-1)\left(1+\rho^2\right)-\frac{(1+B)}{2(1-B)}\\
&\quad-\left[\frac{(1+A)(1+B)}{8\,(A-B)}cx+\frac{(k-1)(1+B)}{4}\right]\rho^2\\
&\quad-\frac{(1-AB)\rho}{4(A-B)}yc+\left[\frac{(1-A)(1-B)cx}{8(A-B)}+\frac{(k-1)\,(B-1)}{4}\right]\\
&=p_1\rho^2+q_1\rho+r_1=Q\,(\rho)\,,
\end{aligned}$$

where

$$\begin{aligned}
p_1&=-\frac{1}{2}\left(k-\frac{1}{2}\right)-\left[\frac{(1+A)(1+B)}{8\,(A-B)}cx+\frac{(k-1)(1+B)}{4}\right],\\
q_1&=-\frac{(1-AB)}{4(A-B)}yc,\\
r_1&=-\frac{1}{2}\left(k-\frac{1}{2}\right)-\frac{(1+B)}{2(1-B)}+\left[\frac{(1-A)(1-B)cx}{8(A-B)}+\frac{(k-1)\,(B-1)}{4}\right].
\end{aligned}$$

Condition (2.1) shows that

$$\begin{aligned}
p_1&=-\frac{1}{2}\left(k-\frac{1}{2}\right)-\left[\frac{(1+A)(1+B)}{8\,(A-B)}cx+\frac{(k-1)(1+B)}{4}\right]\\
&<-\frac{1}{2}\left[\left(k-\frac{1}{2}\right)-\left\{\frac{(1+A)(1+B)}{4\,(A-B)}cx+\frac{(k-1)(1+B)}{2}\right\}\right].
\end{aligned}$$

Since $\max\limits_{\rho\in\mathbb{R}}\{p_1\rho^2+q_1\rho+r_1\}=(4p_1r_1-q_1{}^2)/(4p_1)$ for $p_1<0$, it is clear that $Q(\rho)<0$ when

$$\begin{aligned}
\frac{(1-AB)^2}{16(A-B)^2}y^2c^2&<4\left(-\frac{1}{2}\left(k-\frac{1}{2}\right)-\left[\frac{(1+A)(1+B)}{8\,(A-B)}cx+\frac{(k-1)(1+B)}{4}\right]\right)\\
&\quad\times\left(-\frac{1}{2}\left(k-\frac{1}{2}\right)-\frac{(1+B)}{2(1-B)}+\left[\frac{(1-A)(1-B)cx}{8(A-B)}+\frac{(k-1)\,(B-1)}{4}\right]\right),
\end{aligned}$$

with $|x|,|y|<1$. As $y^2<1-x^2$, the above condition holds whenever

$$\begin{aligned}
\frac{(1-AB)^2}{16(A-B)^2}\left(1-x^2\right)c^2&\le\left[-\left(k-\frac{1}{2}\right)-\frac{(1+A)(1+B)}{4\,(A-B)}cx-\frac{(k-1)(1+B)}{2}\right]\\
&\quad\times\left[-\left(k-\frac{1}{2}\right)-\frac{(1+B)}{(1-B)}+\frac{(1-A)(1-B)cx}{4(A-B)}+\frac{(k-1)\,(B-1)}{2}\right],
\end{aligned}$$

that is, when

$$\frac{c^2}{16}x^2+\left[\left(k-\frac{1}{2}\right)\frac{(A+B)}{2(A-B)}+\frac{(1+A)(1+B)^2}{4(A-B)(1-B)}+\frac{A(1-B^2)}{4(A-B)}(k-1)\right]cx$$
$$+\left[\left(k-\frac{1}{2}\right)^2+\left(k-\frac{1}{2}\right)\frac{(1+B)}{(1-B)}+\frac{(1+B)^2}{(1-B)}\frac{(k-1)}{2}+\left(k-\frac{1}{2}\right)(k-1)\right.$$
$$\left.-\frac{(k-1)^2(B^2-1)}{4}-\frac{(1-AB)^2}{16(A-B)^2}c^2\right]\geq 0. \tag{2.12}$$

To establish inequality (2.12), consider the polynomial R given by

$$R(x):=mx^2+nx+r,\quad |x|<1,$$

where

$$m:=\frac{c^2}{16},$$
$$n:=\left[\left(k-\frac{1}{2}\right)\frac{(A+B)}{2(A-B)}+\frac{(1+A)(1+B)^2}{4(A-B)(1-B)}+\frac{A(1-B^2)}{4(A-B)}(k-1)\right]c,$$
$$r:=\left[\begin{array}{c}\left(k-\frac{1}{2}\right)^2+\left(k-\frac{1}{2}\right)\frac{(1+B)}{(1-B)}+\frac{(1+B)^2}{(1-B)}\frac{(k-1)}{2}\\+\left(k-\frac{1}{2}\right)(k-1)-\frac{(k-1)^2(B^2-1)}{4}-\frac{(1-AB)^2}{16(A-B)^2}c^2\end{array}\right].$$

The constraint (2.3) yields $|n|\geq 2|m|$, and thus $R(x)\geq m+r-|n|$. Now inequality (2.2) readily implies that

$$R(x)\geq m+r-|n|$$
$$=\frac{c^2}{16}+\left[\begin{array}{c}\left(k-\frac{1}{2}\right)^2+\left(k-\frac{1}{2}\right)\frac{(1+B)}{(1-B)}+\frac{(1+B)^2}{(1-B)}\frac{(k-1)}{2}\\+\left(k-\frac{1}{2}\right)(k-1)-\frac{(k-1)^2(B^2-1)}{4}-\frac{(1-AB)^2}{16(A-B)^2}c^2\end{array}\right]$$
$$-\left|\left[\left(k-\frac{1}{2}\right)\frac{(A+B)}{2(A-B)}+\frac{(1+A)(1+B)^2}{4(A-B)(1-B)}+\frac{A(1-B^2)}{4(A-B)}(k-1)\right]c\right|\geq 0.$$

Now considers the case of the constraint (2.5). which is equivalent to $|n|<2m$. Then the minimum of R occurs at $x=-n/(2m)$, and (2.4) yields,

$$R(x)\geq\frac{4mr-n^2}{4m}\geq 0.$$

Evidently Ψ satisfies the hypothesis of Lemma 1.1, and thus $\mathrm{Re}\,p(z)>0$, that is

$$-\frac{(1-A)-(1-B)\,u_v(z)}{(1+A)-(1+B)\,u_v(z)}\prec\frac{1+z}{1-z}.$$

Hence there exists an analytic self-map ω of $\mathcal{U}$ with $\omega(0)=0$ such that

$$-\frac{(1-A)-(1-B)\,u_v(z)}{(1+A)-(1+B)\,u_v(z)}=\frac{1+\omega(z)}{1-\omega(z)},$$

which implies that

$$u_v(z)\prec\frac{1+Az}{1+Bz}.$$

Theorem 2.1 gives rise to simple conditions on c and k to ensure $u_v(z)$ maps $\mathcal{U}$ into a half-plane. □

Corollary 2.2 *Let* $c \le 0$ *and* $4k \ge 3 + c^2$. *Then* $\operatorname{Re} u_v(z) > c/(c-1)$.

Proof Choose $A = -\frac{(c+1)}{(c-1)}$ and $B = -1$ in Theorem 2.1. Then condition (2.1) becomes $k \ge 1/2$. Also condition (2.2) will give

$$\left(k - \frac{1}{2}\right)^2 \ge \frac{\left(k - \frac{1}{2}\right)c^2}{2} - \left(k - \frac{1}{2}\right)(k-1). \tag{2.13}$$

Since $k \ge \frac{1}{4}\left(3 + c^2\right)$, follows that

$$\begin{aligned}\left(k - \frac{1}{2}\right)^2 &+ \left(k - \frac{1}{2}\right)(k-1) - \frac{\left(k - \frac{1}{2}\right)c^2}{2} \\ &= \left(k - \frac{1}{2}\right)\left[\left(k - \frac{1}{2}\right) + (k-1) - \frac{c^2}{2}\right] \ge 0,\end{aligned}$$

which satisfies (2.13). □

Corollary 2.3 *Let* c *and* k *be real such that*

$$k \ge \begin{cases} \frac{1}{2}, & c \le 0 \\ \frac{1}{2} + \frac{c}{4}, & c \ge 0. \end{cases}$$

Then $\operatorname{Re} u_v(z) > 1/2$.

Proof Choose $A = 0$ and $B = -1$ in Theorem 2.1. The condition (2.1) becomes $k \ge 1/2$, which holds in all cases. Condition (2.2) and (2.3) becomes as

$$4\left(k - \frac{1}{2}\right) - c \ge 0, \tag{2.14}$$

and

$$\left(k - \frac{1}{2}\right)^2 + \left(k - \frac{1}{2}\right)(k-1) - \left(k - \frac{1}{2}\right)\frac{c}{2} \ge 0. \tag{2.15}$$

Case 1: For $c \le 0$ both (2.14) and (2.15) holds for $k \ge \frac{1}{2}$.

Case 2: For $c \ge 0$ and $\left(k - \frac{1}{2}\right) \ge \frac{c}{2}$ gives $4\left(k - \frac{1}{2}\right) - c \ge c \ge 0$, and $\left(k - \frac{1}{2}\right)\left[\left(k - \frac{1}{2}\right) + (k-1) - \frac{c}{2}\right] \ge 0$. □

Theorem 2.4 *Let* $3 - \sqrt{2} \le B < A \le 1$ *and* $c, b \in \mathbb{R}$ *satisfy*

$$\left(k - \frac{1}{2}\right) + \frac{(k-1)(1+B)}{2} \ge \max\left\{0, \frac{(1+A)(1+B)}{4(A-B)}|c|\right\}. \tag{2.16}$$

Suppose A, B, b *and* c *satisfy either the inequality*

$$\left[\left(k-\frac{1}{2}\right)^2+16\left(k-\frac{1}{2}\right)\frac{B(1-B)}{(1+B)^3}+\left(k-\frac{1}{2}\right)(k-1)\right.$$
$$\left.-(B^2-1)\frac{(k-1)^2}{4}+\frac{8\,(k-1)\,B(1-B^2)}{(1+B)^3}\right]$$
$$-\left|\left[\left(k-\frac{1}{2}\right)\frac{(A+B)}{2(A-B)}+\frac{4(1+A)(1-B^2)}{(A-B)(1+B)^3}+\frac{A(1-B^2)}{4(A-B)}(k-1)\right]c\right|$$
$$\geq\frac{(1-A^2)\left(1-B^2\right)}{16(A-B)^2}c^2 \tag{2.17}$$

whenever

$$\left|\left[2\left(k-\frac{1}{2}\right)(A+B)\,(1+B)^3+16B(1+A)(1-B^2)+A(1-B^2)\,(1+B)^3\,(k-1)\right]c\right|$$
$$\geq\frac{c^2}{2}\,(A-B)\,(1+B)^3\,, \tag{2.18}$$

or the inequality

$$\left[\left(k-\frac{1}{2}\right)\frac{(A+B)}{2(A-B)}+\frac{4(1+A)(1-B^2)}{(A-B)(1+B)^3}+\frac{A(1-B^2)}{4(A-B)}(k-1)\right]^2c^2$$
$$\leq\frac{c^2}{16}\left[\begin{array}{c}\left(k-\frac{1}{2}\right)^2+16\left(k-\frac{1}{2}\right)\frac{B(1-B)}{(1+B)^3}+\left(k-\frac{1}{2}\right)(k-1)-(B^2-1)\frac{(k-1)^2}{4}\\ +\frac{8(k-1)B(1-B^2)}{(1+B)^3}-\frac{(1-AB)^2}{16(A-B)^2}c^2\end{array}\right] \tag{2.19}$$

whenever

$$\left|\left[2\left(k-\frac{1}{2}\right)(A+B)\,(1+B)^3+16B(1+A)(1-B^2)+A(1-B^2)\,(1+B)^3\,(k-1)\right]c\right|$$
$$<\frac{c^2}{2}\,(A-B)\,(1+B)^3\,. \tag{2.20}$$

If $(1+B)u_v(z)\neq(1+A)$*, then* $u_v(z)\in\mathcal{P}[A,B]$.

Proof First, proceed similar to the proof Theorem 2.1 and derive the expression of $\mathrm{Re}\Psi(i\rho,\sigma,\mu+i\nu;z)$ as given in (2.11). Now for $\sigma\leq-(1+\rho^2)/2$, $\rho\in\mathbb{R}$, and $B\geq 3-\sqrt{2}$,

$$\frac{2\left(1-B^2\right)}{(1-B)^2+(1+B)^2\rho^2}\sigma^2\geq\frac{2\left(1-B^2\right)}{(1-B)^2+(1+B)^2\rho^2}\left(\frac{1+\rho^2}{2}\right)^2\geq\frac{8B\,(1-B)}{(1+B)^3},$$

and then with $z = x + iy \in \mathcal{U}$, and $\mu + \sigma < 0$, it follow that

$$\begin{aligned}
&\operatorname{Re} \Psi(i\rho, \sigma, \mu + i\nu; z) \\
&\le \{\mu + \sigma\} - \frac{1}{4}(2k-1)\left(1+\rho^2\right) - \frac{8B(1-B)}{(1+B)^3} \\
&\quad - \left[\frac{(1+A)(1+B)}{8(A-B)}cx + \frac{(k-1)(1+B)}{4}\right]\rho^2 \\
&\quad - \frac{(1-AB)\rho}{4(A-B)}yc + \left[\frac{(1-A)(1-B)cx}{8(A-B)} + \frac{(k-1)(B-1)}{4}\right] \\
&= p_2\rho^2 + q_2\rho + r_2 = Q_1(\rho),
\end{aligned}$$

where

$$\begin{aligned}
p_2 &= -\frac{1}{2}\left(k - \frac{1}{2}\right) - \left[\frac{(1+A)(1+B)}{8(A-B)}cx + \frac{(k-1)(1+B)}{4}\right], \\
q_2 &= -\frac{(1-AB)}{4(A-B)}yc, \\
r_2 &= -\frac{1}{2}\left(k - \frac{1}{2}\right) - \frac{8B(1-B)}{(1+B)^3} + \left[\frac{(1-A)(1-B)cx}{8(A-B)} + \frac{(k-1)(B-1)}{4}\right].
\end{aligned}$$

Observe that the inequality (2.16) implies that $p_2 < 0$. Thus $Q_1(\rho) < 0$ for all $\rho \in \mathbb{R}$ provided $q_2^2 \le 4p_2r_2$, that is, for $|x|, |y| < 1$,

$$\begin{aligned}
\frac{(1-AB)^2}{16(A-B)^2}y^2c^2 &< 4\left(-\frac{1}{2}\left(k-\frac{1}{2}\right) - \left[\frac{(1+A)(1+B)}{8(A-B)}cx + \frac{(k-1)(1+B)}{4}\right]\right) \\
&\quad \times \left(-\frac{1}{2}\left(k-\frac{1}{2}\right) - \frac{8B(1-B)}{(1+B)^3} + \frac{(1-A)(1-B)cx}{8(A-B)} + \frac{(k-1)(B-1)}{4}\right),
\end{aligned}$$

with $y^2 < 1 - x^2$, it is enough to show for $|x| < 1$,

$$\begin{aligned}
&\frac{(1-AB)^2}{16(A-B)^2}\left(1-x^2\right)c^2 < 4\left(-\frac{1}{2}\left(k-\frac{1}{2}\right) - \left[\frac{(1+A)(1+B)}{8(A-B)}cx + \frac{(k-1)(1+B)}{4}\right]\right) \\
&\times \left(-\frac{1}{2}\left(k-\frac{1}{2}\right) - \frac{8B(1-B)}{(1+B)^3} + \frac{(1-A)(1-B)cx}{8(A-B)} + \frac{(k-1)(B-1)}{4}\right),
\end{aligned}$$

which is equivalent to

$$R(x) := m_1x^2 + n_1x + r_1 \ge 0, \tag{2.21}$$

where,

$$\begin{aligned}
m_1 &:= \frac{c^2}{16}, \\
n_1 &:= \left[\left(k-\frac{1}{2}\right)\frac{(A+B)}{2(A-B)} + \frac{4B(1+A)(1-B^2)}{(A-B)(1+B)^3} + \frac{A(1-B^2)}{4(A-B)}(k-1)\right]c, \\
r_1 &:= \left[\begin{array}{c}\left(k-\frac{1}{2}\right)^2 + 16\left(k-\frac{1}{2}\right)\frac{B(1-B)}{(1+B)^3} + \left(k-\frac{1}{2}\right)(k-1) - (B^2-1)\frac{(k-1)^2}{4} \\ + \frac{8(k-1)B(1-B^2)}{(1+B)^3} - \frac{(1-AB)^2}{16(A-B)^2}c^2\end{array}\right].
\end{aligned}$$

If (2.18) holds, then $|n_1| \ge 2|m_1|$. Since $R_1(x) \ge m_1 + r_1 - |n_1|$, which is non-negative from (2.17). On the other hand, if (2.20) holds, then $|n_1| < 2|m_1|$, $R_1(x) \ge (4m_1r_1 - n_1^2)/4m$, and implies $R_1(x) \ge 0$. Either case establishes (2.21). □

Reasoning along the same lines as the proof of the Theorem 2.1 we obtain the following theorems. We omit the details.

Theorem 2.5 *Let* $-1 \leq B \leq 3-\sqrt{2} \approx 0.171573$.*Suppose* $B < A \leq 1$, $b, c \in \mathbb{R}$, $c \neq 0$, *and satisfying*

$$\left(k+\frac{1}{2}\right)+\frac{k(1+B)}{2} \geq \max\left\{0, \frac{(1+A)\,(1+B)}{4\,(A-B)}\,|c|\right\}. \tag{2.22}$$

Suppose that A, B, b *and* c *satisfy either*

$$\begin{aligned}&\left[\left(k+\tfrac{1}{2}\right)^2+\left(k+\tfrac{1}{2}\right)\tfrac{(1+B)}{(1-B)}+k\tfrac{(1+B)^2}{2(1-B)}+k\left(k+\tfrac{1}{2}\right)-\tfrac{k^2(B^2-1)}{4}\right]\\&\quad-\left|\left(k+\tfrac{1}{2}\right)\tfrac{(A+B)}{2(A-B)}+\tfrac{(1+A)(1+B)^2}{4(A-B)(1-B)}+\tfrac{A(1-B^2)}{4(A-B)}k\right| \geq \tfrac{(1-A^2)\left(1-B^2\right)}{16(A-B)^2}c^2,\end{aligned} \tag{2.23}$$

whenever

$$\begin{aligned}&\left|\left[2\left(k+\frac{1}{2}\right)(A+B)(1-B)+(1+A)(1+B)^2+A(1-B^2)(1-B)k\right]c\right|\\&\quad\geq \frac{1}{2}\,(A-B)\,(1-B)\,c^2.\end{aligned} \tag{2.24}$$

or the inequality

$$\begin{aligned}&\left[\left(k+\frac{1}{2}\right)\frac{(A+B)}{2(A-B)}+\frac{(1+A)(1+B)^2}{4(A-B)(1-B)}+\frac{A(1-B^2)}{4(A-B)}k\right]^2c^2\\&\quad\leq \frac{c^2}{16}\left[\left(k+\frac{1}{2}\right)^2+\left(k+\frac{1}{2}\right)\frac{(1+B)}{(1-B)}+k\frac{(1+B)^2}{2(1-B)}\right.\\&\qquad\left.+k\left(k+\frac{1}{2}\right)-\frac{k^2(B^2-1)}{4}-\frac{(1-AB)^2}{16(A-B)^2}c^2\right],\end{aligned} \tag{2.25}$$

whenever

$$\begin{aligned}&\left|\left[2\left(k+\frac{1}{2}\right)(A+B)(1-B)+(1+A)(1+B)^2+A(1-B^2)(1-B)k\right]c\right|\\&\quad< \frac{1}{2}\,(A-B)\,(1-B)c^2.\end{aligned} \tag{2.26}$$

If $(1+B)\,u_{v+1}\,(z) \neq (1+A)$ *and* $cz \neq 0$, *then* $\frac{2k}{cz}\left[1-2u_v'\,(z)-u_v(z)\right] \in \mathcal{P}\,[A,B]$.

Theorem 2.6 *Let* $3-\sqrt{2} < B < A \leq 1$. *Suppose* $b,\ c \in \mathbb{R}$, $a \neq 0$, *such that*

$$\left(k+\frac{1}{2}\right)+\frac{k(1+B)}{2} \geq \max\left\{0, \frac{(1+A)\,(1+B)}{4\,(A-B)}\,|c|\right\}.$$

Suppose that A, B, b *and* c *satisfy either*

$$\begin{aligned}&\left[\left(k+\frac{1}{2}\right)^2+k\left(k+\frac{1}{2}\right)-\frac{k^2}{4}(B^2-1)+16(k+\frac{1}{2})\frac{B(1-B)}{(1+B)^3}+8k\frac{B(1-B)^2}{(1+B)^3}\right]\\&\quad-\left|\left[\left(k+\frac{1}{2}\right)\frac{(A+B)}{2(A-B)}+4\frac{(1+A)(1-B^2)B}{(A-B)(1+B)^3}-\frac{A(1-B^2)}{4(A-B)}k\right]c\right|\\&\geq \frac{(1-A^2)\left(1-B^2\right)}{16(A-B)^2}c^2\end{aligned}$$

whenever

$$\left|\left[2\left(k+\frac{1}{2}\right)(A+B)(1+B)^3+16(1+A)(1-B^2)B+A(1-B^2)(1+B)^3k\right]c\right| \geq \frac{1}{2}c^2(A-B)(1+B)^3,$$

or the inequality

$$\left[\left(k+\frac{1}{2}\right)\frac{(A+B)c}{2(A-B)}+4\frac{(1+A)(1-B^2)Bc}{(A-B)(1+B)^3}-\frac{A(1-B^2)}{4(A-B)}kc\right]^2 c^2 \leq \frac{c^2}{16}\left[\left(k+\frac{1}{2}\right)^2+k\left(k+\frac{1}{2}\right)-\frac{k^2}{4}(B^2-1)+16(k+\frac{1}{2})\frac{B(1-B)}{(1+B)^3}+8k\frac{B(1-B)^2}{(1+B)^3}\right],$$

whenever

$$\left|\left[2\left(k+\frac{1}{2}\right)(A+B)(1+B)^3+16(1+A)(1-B^2)B+A(1-B^2)(1+B)^3k\right]c\right| < \frac{1}{2}c^2(A-B)(1+B)^3.$$

If $(1+B)\,u_{v+1}(z) \neq (1+A)$ *and* $cz \neq 0$, *then* $\frac{2k}{cz}\left[1-2u_v'(z)-u_v(z)\right] \in \mathcal{P}[A,B]$.

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