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ORIGINAL PAPER

Radii of Uniform Convexity of Lommel and Struve Functions

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Abstract
In this paper, we determine the radii of β-uniform convexity of order α for six kinds of normalized Lommel and Struve functions of the first kind. One of the most important things which we have learned in this study is that the radii of uniform convexity are obtained as solutions of some transcendental equations.

Keywords Lommel functions · Struve functions · Univalent functions · β-uniformly convex functions of order α · Zeros of Lommel functions of the first kind · Zeros of Struve functions of the first kind

Mathematics Subject Classification 30C45 · 30C80 · 33C10

1 Introduction and Preliminaries

It is well known that the concepts of convexity, starlikeness, close-to-convexity and uniform convexity including necessary and sufficient conditions, have a long history as a part of geometric function theory. It is known that special functions, like Bessel, Struve and Lommel functions of the first kind have some beautiful geometric properties. Recently, the above geometric properties of the Bessel functions were investigated in some earlier results (see [1–5,12]). On the other hand, the radii of convexity and starlikeness of the Struve and Lommel functions were studied by Baricz et al. [8,10]. Motivated by the above developments, in this paper, our aim was

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to give some new results for the radius of β-uniformly convex functions of order α of the normalized Struve and Lommel functions of the first kind. In the special cases of the parameters α and β, we can obtain some earlier results. The key tools were some Mittag–Leffler expansions of Lommel and Struve functions of the first kind and special properties of the zeros of these functions and their derivatives. Let $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ denote the disk of radius r and center z_0. We use $U(r) = U(0, r)$ and $U = U(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$. Let $(a_n)_{n\geq 2}$ be a sequence of complex numbers with

$$d = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \geq 0, \text{ and } r_f = \frac{1}{d}.$$

If $d = 0$, then $r_f = +\infty$. As usual, we denote by $\mathcal{A}$ the class of all analytic functions $f : U(r_f) \to \mathbb{C}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We say that a function f of the form (1.1) is convex if f is univalent and $f(U(r))$ is a convex domain in $\mathbb{C}$. An analytic description of this definition is that

$$f \in \mathcal{A} \text{ is convex if and only if } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U(r).$$

The radius of convexity of the function f is defined by

$$r_f^c = \sup\left\{r \in (0, r_f) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U(r)\right\}.$$

In the following, we deal with the class of the uniformly convex functions. Goodman in [13] introduced the concept of uniform convexity for functions of the form (1.1). A function f is said to be uniformly convex in $U(r)$ if f is of the form (1.1), it is convex, and has the property that for every circular arc γ contained in $U(r)$, with center ς, the arc $f(\gamma)$ is convex. In 1993, Rønning [21] determined necessary and sufficient conditions of analytic functions to be uniformly convex in the open unit disk, while in 2002 Ravichandran [20] also presented simpler criteria for uniform convexity. Rønning in [21] gives an analytic description of the uniformly convex functions in the following theorem:

Theorem 1.1 *Let f be a function of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the disk $U(r)$. The function f is uniformly convex in the disk $U(r)$ if and only if*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U(r).$$

The class of uniformly convex function is denoted by UC. The radius of uniform convexity is defined by

$$r_f^{uc} = \sup \left\{ r \in (0, r_f) : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in U(r) \right\}.$$

In 1997, Bharti, Parvatham and Swaminathan defined β-uniformly convex functions of order α which is a subclass of uniformly convex functions. A function $f \in \mathcal{A}$ is said to be in the class of β-uniformly convex functions of order α, denoted by β-$UC(\alpha)$, if

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta \left| \frac{zf''(z)}{f'(z)} \right| + \alpha,$$

where $\beta \geq 0, \ \alpha \in [0, 1)$ (see [11]).

This class generalizes various other classes which are worthy to mention here. For example, the class β-$UC(0) = \beta$-UC is the class of β-uniformly convex functions [14] (also see [15] and [16]) and 1-$UC(0) = UC$ is the class of uniformly convex functions defined by Goodman [13] (see also [21]).

Geometric Interpretation. It is known that $f \in \beta$-$UC(\alpha)$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ takes all the values in the conic domain $\mathcal{R}_{\beta,\alpha}$ which is included in the right half plane given by

$$\mathcal{R}_{\beta,\alpha} := \left\{ w = u + iv \in \mathbb{C} : u > \beta\sqrt{(u-1)^2 + v^2} + \alpha, \ \beta \geq 0 \text{ and } \alpha \in [0, 1) \right\}.$$

Then, $\partial\mathcal{R}_{\beta,\alpha}$ is a curve defined by

$$\partial\mathcal{R}_{\beta,\alpha} := \left\{ w = u + iv \in \mathbb{C} : u^2 = \left(\beta\sqrt{(u-1)^2 + v^2} + \alpha \right)^2, \ \beta \geq 0 \text{ and } \alpha \in [0, 1) \right\}. \tag{1.2}$$

From elementary computations, we see that (1.2) represents conic sections symmetric about the real axis. Thus, $\mathcal{R}_{\beta,\alpha}$ is an elliptic domain for $\beta > 1$, a parabolic domain for $\beta = 1$, a hyperbolic domain for $0 < \beta < 1$ and the right half plane $u > \alpha$, for $\beta = 0$.

The radius of β-uniform convexity of order α is defined by

$$r_f^{\beta\text{-}uc(\alpha)} = \sup \left\{ r \in (0, r_f) : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta \left| \frac{zf''(z)}{f'(z)} \right| + \alpha, \ \beta \geq 0, \ \alpha \in [0, 1), \ z \in U(r) \right\}.$$

Our main aim was to determine the radii of β-uniform convexity of order α of Lommel and Struve functions.

In order to prove the main results, we need the following lemma given in [12]:

Lemma 1.2 i. *If* $a > b > r \geq |z|$ *and* $\lambda \in [0, 1]$, *then*

$$\left| \frac{z}{b-z} - \lambda \frac{z}{a-z} \right| \leq \frac{r}{b-r} - \lambda \frac{r}{a-r}. \tag{1.3}$$

In particular,

$$\Re\left(\frac{z}{b-z}-\lambda\frac{z}{a-z}\right)\le\frac{r}{b-r}-\lambda\frac{r}{a-r} \tag{1.4}$$

and

$$\Re\left(\frac{z}{b-z}\right)\le\left|\frac{z}{b-z}\right|\le\frac{r}{b-r}. \tag{1.5}$$

ii. *If* $b>a>r\ge|z|$, *then*

$$\left|\frac{1}{(a+z)(b-z)}\right|\le\frac{1}{(a-r)(b+r)}.$$

2 Main Results

In this paper our aim was to consider two classical special functions, the Lommel function of the first kind $s_{\mu,\nu}$ and the Struve function of the first kind $\mathbf{H}_\nu$. They are explicitly defined in terms of the hypergeometric function ${}_1F_2$ by

$$s_{\mu,\nu}(z)=\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left(1;\frac{(\mu-\nu+3)}{2},\frac{(\mu+\nu+3)}{2};-\frac{z^2}{4}\right),$$

$\frac{1}{2}(-\mu\pm\nu-3)\notin\mathbb{N}$ and

$$\mathbf{H}_\nu(z)=\frac{\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\frac{\pi}{4}}\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_1F_2\left(1;\frac{3}{2},\nu+\frac{3}{2};-\frac{z^2}{4}\right),\quad -\nu-\frac{3}{2}\notin\mathbb{N}.$$

Observe that

$$s_{\nu,\nu}(z)=2^{\nu-1}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_\nu(z).$$

It is well known that they are solutions of inhomogeneous Bessel differential equations [24]. Indeed, the Lommel function of the first kind $s_{\mu,\nu}$ is a solution of

$$z^2w''(z)+zw'(z)+\left(z^2-\nu^2\right)w(z)=z^{\mu+1},$$

while the Struve function $\mathbf{H}_\nu$ obeys

$$z^2w''(z)+zw'(z)+\left(z^2-\nu^2\right)w(z)=\frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}.$$

We refer to Watson's treatise [24] for comprehensive information about these functions and recall here briefly some contributions. In 1970, Steinig [22] investigated the real zeros of the Struve function $\mathbf{H}_\nu$, while in 1972, he [23] examined the sign of $s_{\mu,\nu}(z)$ for real μ,ν and positive z. He showed, among other things, that for $\mu<\frac{1}{2}$, the function $s_{\mu,\nu}$ has infinitely many changes of sign on $(0,\infty)$. In 2012, Koumandos and Lamprecht [17] obtained sharp estimates for the location of the zeros of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ when $\mu \in (0,1)$. Turán type inequalities for $s_{\mu-\frac{1}{2},\frac{1}{2}}$ were established in [6] while those for Struve function were proved in [9]. Geometric properties of the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and the Struve function $\mathbf{H}_\nu$ were obtained in [7,8,10,19,25]. Motivated by those results, in this paper we are interested in the radii of β-uniformly convex of order α of following functions. Since neither $s_{\mu-\frac{1}{2},\frac{1}{2}}$, nor $\mathbf{H}_\nu$ belongs to the class analytic functions, first we perform some natural normalizations, as in [8]. Three functions related to $s_{\mu-\frac{1}{2},\frac{1}{2}}$ can be defined as follows:

$$f_\mu(z) = f_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \Big(\mu\,(\mu+1)\,s_{\mu-\frac{1}{2},\frac{1}{2}}(z)\Big)^{\frac{1}{\mu+\frac{1}{2}}},$$
$$g_\mu(z) = g_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \mu\,(\mu+1)\,z^{-\mu+\frac{1}{2}}s_{\mu-\frac{1}{2},\frac{1}{2}}(z)$$

and

$$h_\mu(z) = h_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \mu\,(\mu+1)\,z^{\frac{3-2\mu}{4}}s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{z}).$$

Similarly, we associate with $\mathbf{H}_\nu$ the functions

$$u_\nu(z) = \left(\sqrt{\pi}2^\nu\,\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)\right)^{\frac{1}{\nu+1}},$$
$$v_\nu(z) = \sqrt{\pi}2^\nu z^{-\nu}\,\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)$$

and

$$w_\nu(z) = \sqrt{\pi}2^\nu z^{\frac{1-\nu}{2}}\,\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu\left(\sqrt{z}\right).$$

Clearly, the functions f_μ, g_μ, h_μ, u_ν, v_ν and w_ν belong to the class of analytic functions $\mathcal{A}$. The main results in the present paper concern some exact values of the radii of β-uniform convexity of order α for above six functions, for some ranges of the parameters.

In the following, we present some lemmas given by Baricz and Yağmur [10], on the zeros of derivatives of Lommel and Struve functions of the first kind. These lemmas are the key tools in the proof of our main results.

Lemma 2.1 [10] *The zeros of the Lommel function* $s_{\mu-\frac{1}{2},\frac{1}{2}}$ *and its derivative interlace when* $\mu \in (-1,1)$, $\mu \neq 0$. *Moreover, the zeros* $\xi'_{\mu,n}$ *of the function* $s'_{\mu-\frac{1}{2},\frac{1}{2}}$ *are all real and simple when* $\mu \in (-1,1)$, $\mu \neq 0$.

Lemma 2.2 [10] *The zeros of the function* $\mathbf{H}_\nu$ *and its derivative interlace when* $|\nu| \leq \frac{1}{2}$. *Moreover, the zeros* $h'_{\nu,n}$ *of the function* $\mathbf{H}'_\nu$ *are all real and simple when* $|\nu| \leq \frac{1}{2}$.

Now, the first main result of this section presents the radii of β-uniform convexity of order α of functions f_μ, g_μ and h_μ.

Theorem 2.3 *Let* $\mu \in (0, 1)$, $\beta \geq 0$ *and* $\alpha \in [0, 1)$. *Then, the radius of* β*-uniform convexity of order* α *of the function* f_μ *is the smallest positive root of the equation*

$$(1-\alpha)+(1+\beta)\left(\frac{rs''_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{s'_{\mu-\frac{1}{2},\frac{1}{2}}(r)}+\left(\frac{1}{\mu+\frac{1}{2}}-1\right)\frac{rs'_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{s_{\mu-\frac{1}{2},\frac{1}{2}}(r)}\right)=0.$$

Moreover, $r^{\beta\text{-}uc(\alpha)}_{f_\mu} < r^c_{f_\mu} < \xi'_{\mu,1} < \xi_{\mu,1}$, *where* $\xi_{\mu,1}$ *and* $\xi'_{\mu,1}$ *denote the first positive zeros of* $s_{\mu-\frac{1}{2},\frac{1}{2}}$ *and* $s'_{\mu-\frac{1}{2},\frac{1}{2}}$, *respectively, and* $r^c_{f_\mu}$ *is the radius of convexity of the function* f_μ.

Proof In [10], authors proved the Mittag–Leffler expansions of $s_{\mu-\frac{1}{2},\frac{1}{2}}(z)$ and $s'_{\mu-\frac{1}{2},\frac{1}{2}}(z)$ as follows:

$$s_{\mu-\frac{1}{2},\frac{1}{2}}(z)=\frac{z^{\mu+\frac{1}{2}}}{\mu(\mu+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\xi^2_{\mu,n}}\right) \tag{2.1}$$

and

$$s'_{\mu-\frac{1}{2},\frac{1}{2}}(z)=\frac{\left(\mu+\frac{1}{2}\right)z^{\mu-\frac{1}{2}}}{\mu(\mu+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\xi'^2_{\mu,n}}\right), \tag{2.2}$$

where $\xi_{\mu,n}$ and $\xi'_{\mu,n}$ denote the n-th positive roots of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and $s'_{\mu-\frac{1}{2},\frac{1}{2}}$, respectively. Observe also that

$$1+\frac{zf''_\mu(z)}{f'_\mu(z)}=1+\frac{zs''_{\mu-\frac{1}{2},\frac{1}{2}}(z)}{s'_{\mu-\frac{1}{2},\frac{1}{2}}(z)}+\left(\frac{1}{\mu+\frac{1}{2}}-1\right)\frac{zs'_{\mu-\frac{1}{2},\frac{1}{2}}(z)}{s_{\mu-\frac{1}{2},\frac{1}{2}}(z)}. \tag{2.3}$$

Thus, from (2.1), (2.2) and (2.3), we have

$$1+\frac{zf''_\mu(z)}{f'_\mu(z)}=1-\left(\frac{1}{\mu+\frac{1}{2}}-1\right)\sum_{n\geq 1}\frac{2z^2}{\xi^2_{\mu,n}-z^2}-\sum_{n\geq 1}\frac{2z^2}{\xi'^2_{\mu,n}-z^2}.$$

Now, the proof will be presented in two cases by considering the intervals of μ. First, suppose that $\mu \in \left(0, \frac{1}{2}\right]$. Since $\frac{1}{\mu+\frac{1}{2}}-1 \geq 0$, inequality (1.5) implies for $|z| \leq r < \xi'_{\mu,1} < \xi_{\mu,1}$

$$\Re\left(1+\frac{zf''_\mu(z)}{f'_\mu(z)}\right)=1-\sum_{n\geq 1}\Re\left(\frac{2z^2}{\xi'^2_{\mu,n}-z^2}\right)-\left(\frac{1}{\mu+\frac{1}{2}}-1\right)\sum_{n\geq 1}\Re\left(\frac{2z^2}{\xi^2_{\mu,n}-z^2}\right)$$

$$\begin{aligned}&\geq 1-\sum_{n\geq 1}\frac{2r^2}{\xi_{\mu,n}'^2-r^2}-\left(\frac{1}{\mu+\frac{1}{2}}-1\right)\sum_{n\geq 1}\frac{2r^2}{\xi_{\mu,n}^2-r^2}\\ &=1+\frac{rf_\mu''(r)}{f_\mu'(r)}.\end{aligned}\tag{2.4}$$

On the other hand, if in the second part of inequality (1.5) we replace z by z^2 and b by $\xi'_{\mu,n}$ and $\xi_{\mu,n}$, respectively, then it follows that

$$\left|\frac{2z^2}{\xi_{\mu,n}'^2-z^2}\right|\leq\frac{2r^2}{\xi_{\mu,n}'^2-r^2}\text{ and }\left|\frac{2z^2}{\xi_{\mu,n}^2-z^2}\right|\leq\frac{2r^2}{\xi_{\mu,n}^2-r^2},$$

provided that $|z|\leq r<\xi'_{\mu,1}<\xi_{\mu,1}$. These two inequalities and the conditions $\frac{1}{\mu+\frac{1}{2}}-1\geq 0$ and $\beta\geq 0$ imply that

$$\begin{aligned}\beta\left|\frac{zf_\mu''(z)}{f_\mu'(z)}\right|&=\beta\left|\sum_{n\geq 1}\left(\frac{2z^2}{\xi_{\mu,n}'^2-z^2}+\left(\frac{1}{\mu+\frac{1}{2}}-1\right)\frac{2z^2}{\xi_{\mu,n}^2-z^2}\right)\right|\\ &\leq\beta\sum_{n\geq 1}\left|\frac{2z^2}{\xi_{\mu,n}'^2-z^2}\right|+\beta\left(\frac{1}{\mu+\frac{1}{2}}-1\right)\sum_{n\geq 1}\left|\frac{2z^2}{\xi_{\mu,n}^2-z^2}\right|\\ &\leq\beta\sum_{n\geq 1}\left(\frac{2r^2}{\xi_{\mu,n}'^2-r^2}+\left(\frac{1}{\mu+\frac{1}{2}}-1\right)\frac{2r^2}{\xi_{\mu,n}^2-r^2}\right)=-\beta\frac{rf_\mu''(r)}{f_\mu'(r)}.\end{aligned}\tag{2.5}$$

From (2.4) and (2.5), we infer

$$\Re\left(1+\frac{zf_\mu''(z)}{f_\mu'(z)}\right)-\beta\left|\frac{zf_\mu''(z)}{f_\mu'(z)}\right|-\alpha\geq 1-\alpha+(1+\beta)\frac{rf_\mu''(r)}{f_\mu'(r)},$$

where $|z|\leq r<\xi'_{\mu,1},\ \beta\geq 0$ and $\alpha\in[0,1)$.

In the second step we will prove that inequalities (2.4) and (2.5) hold in the case $\mu\in\left(\frac{1}{2},1\right)$, too. Indeed, in the case $\mu\in\left(\frac{1}{2},1\right)$, the roots $0<\xi'_{\mu,n}<\xi_{\mu,n}$ are real for every natural number n. Moreover, inequality (1.5) implies that

$$\Re\left(\frac{2z^2}{\xi_{\mu,n}'^2-z^2}\right)\leq\left|\frac{2z^2}{\xi_{\mu,n}'^2-z^2}\right|\leq\frac{2r^2}{\xi_{\mu,n}'^2-r^2},\quad |z|\leq r<\xi'_{\mu,1}<\xi_{\mu,1}$$

and

$$\Re\left(\frac{2z^2}{\xi_{\mu,n}^2-z^2}\right)\leq\left|\frac{2z^2}{\xi_{\mu,n}^2-z^2}\right|\leq\frac{2r^2}{\xi_{\mu,n}^2-r^2},\quad |z|\leq r<\xi'_{\mu,1}<\xi_{\mu,1}.$$

Putting $\lambda = 1 - \frac{1}{\mu+\frac{1}{2}}$ inequality (1.4) implies

$$\Re\left(\frac{2z^2}{\xi'^2_{\mu,n} - z^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2z^2}{\xi^2_{\mu,n} - z^2}\right) \le \frac{2r^2}{\xi'^2_{\mu,n} - r^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2r^2}{\xi^2_{\mu,n} - r^2},$$

for $|z| \le r < \xi'_{\mu,1} < \xi_{\mu,1}$, and we get

$$\begin{aligned}\Re\left(1 + \frac{zf''_\mu(z)}{f'_\mu(z)}\right) &= 1 - \sum_{n\ge1}\Re\left(\frac{2z^2}{\xi'^2_{\mu,n} - z^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2z^2}{\xi^2_{\mu,n} - z^2}\right)\\ &\ge 1 - \sum_{n\ge1}\left(\frac{2r^2}{\xi'^2_{\mu,n} - r^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2r^2}{\xi^2_{\mu,n} - r^2}\right)\\ &= 1 + \frac{rf''_\mu(r)}{f'_\mu(r)}. \end{aligned} \tag{2.6}$$

Now, if in the inequality (1.3), we replace z by z^2 and b by $\xi'_{\mu,n}$ and $\xi_{\mu,n}$ we again put $\lambda = 1 - \frac{1}{\mu+\frac{1}{2}}$, it follows that

$$\left|\frac{2z^2}{\xi'^2_{\mu,n} - z^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2z^2}{\xi^2_{\mu,n} - z^2}\right| \le \frac{2r^2}{\xi'^2_{\mu,n} - r^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2r^2}{\xi^2_{\mu,n} - r^2},$$

provided that $|z| \le r < \xi'_{\mu,1} < \xi_{\mu,1}$. Thus, for $\beta \ge 0$ we obtain

$$\begin{aligned}\beta\left|\frac{zf''_\mu(z)}{f'_\mu(z)}\right| &= \beta\left|\sum_{n\ge1}\left(\frac{2z^2}{\xi'^2_{\mu,n} - z^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2z^2}{\xi^2_{\mu,n} - z^2}\right)\right|\\ &\le \beta\sum_{n\ge1}\left|\frac{2z^2}{\xi'^2_{\mu,n} - z^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2z^2}{\xi^2_{\mu,n} - z^2}\right|\\ &\le \beta\sum_{n\ge1}\left(\frac{2r^2}{\xi'^2_{\mu,n} - r^2} - \left(1 - \frac{1}{\mu + \frac{1}{2}}\right)\frac{2r^2}{\xi^2_{\mu,n} - r^2}\right) = -\beta\frac{rf''_\mu(r)}{f'_\mu(r)}. \end{aligned} \tag{2.7}$$

Finally, the following inequality be inferred from (2.6) and (2.7) for $\mu \in \left(\frac{1}{2}, 1\right)$,

$$\Re\left(1 + \frac{zf''_\mu(z)}{f'_\mu(z)}\right) - \beta\left|\frac{zf''_\mu(z)}{f'_\mu(z)}\right| - \alpha \ge 1 - \alpha + (1+\beta)\frac{rf''_\mu(r)}{f'_\mu(r)},$$

where $|z| \le r < \xi'_{\mu,1}$, $\beta \ge 0$ and $\alpha \in [0, 1)$.

Equality holds (2.6) if and only if $z = r$. Thus, it follows that

$$\inf_{|z|<r}\left[\Re\left(1+\frac{zf_\mu''(z)}{f_\mu'(z)}\right)-\beta\left|\frac{zf_\mu''(z)}{f_\mu'(z)}\right|-\alpha\right]=1-\alpha+(1+\beta)\frac{rf_\mu''(r)}{f_\mu'(r)},$$

for $r \in (0, \xi'_{\mu,1})$, $\alpha \in [0, 1)$, $\beta \geq 0$ and $\mu \in (0, 1)$.

The mapping $\psi_\mu : (0, \xi'_{\mu,1}) \to \mathbb{R}$ defined by

$$\psi_\mu(r)=1-\alpha+(1+\beta)\frac{rf_\mu''(r)}{f_\mu'(r)}=1-\alpha-(1+\beta)\sum_{n\geq 1}\left(\frac{2r^2}{\xi'^2_{\mu,n}-r^2}-\left(1-\frac{1}{\mu+\frac{1}{2}}\right)\frac{2r^2}{\xi^2_{\mu,n}-r^2}\right)$$

is strictly decreasing for all $\mu \in (0, 1)$, $\beta \geq 0$ and $\alpha \in [0, 1)$. Namely, we obtain

$$\begin{aligned}\psi_\mu'(r)&=-(1+\beta)\sum_{n\geq 1}\left(\frac{4r^2\xi'^2_{\mu,n}}{\left(\xi'^2_{\mu,n}-r^2\right)^2}-\left(1-\frac{1}{\mu+\frac{1}{2}}\right)\frac{4r^2\xi^2_{\mu,n}}{\left(\xi^2_{\mu,n}-r^2\right)^2}\right)\\&<(1+\beta)\sum_{n\geq 1}\left(\frac{4r^2\xi^2_{\mu,n}}{\left(\xi^2_{\mu,n}-r^2\right)^2}-\frac{4r^2\xi'^2_{\mu,n}}{\left(\xi'^2_{\mu,n}-r^2\right)^2}\right)<0\end{aligned}$$

for $\mu \in (\frac{1}{2}, 1)$, $r \in (0, \xi'_{\mu,1})$ and $\beta \geq 0$. Here, we used again that the zeros $\xi_{\mu,n}$ and $\xi'_{\mu,n}$ interlace, and for all $n \in \mathbb{N}$, $\mu \in (0, 1)$ and $r < \sqrt{\xi_{\mu,n}\xi'_{\mu,n}}$, we have that

$$\xi^2_{\mu,n}\left(\xi'^2_{\mu,n}-r^2\right)^2<\xi'^2_{\mu,n}\left(\xi^2_{\mu,n}-r^2\right)^2.$$

Let now $\mu \in \left(0, \frac{1}{2}\right]$ and $r > 0$. Thus, the following inequality

$$\psi_\mu'(r)=-(1+\beta)\sum_{n\geq 1}\left(\frac{4r^2\xi'^2_{\mu,n}}{\left(\xi'^2_{\mu,n}-r^2\right)^2}-\left(1-\frac{1}{\mu+\frac{1}{2}}\right)\frac{4r^2\xi^2_{\mu,n}}{\left(\xi^2_{\mu,n}-r^2\right)^2}\right)<0$$

is satisfied and thus ψ_μ is indeed strictly decreasing for all $\mu \in (0, 1)$ and $\beta \geq 0$.

Now, since $\lim_{r\searrow 0}\psi_\mu(r) = 1$ and $\lim_{r\nearrow \xi'_{\mu,1}}\psi_\mu(r) = -\infty$, in view of the minimum principle for harmonic functions, it follows that for $\mu \in (0, 1)$ and $z \in U(r^{\beta\text{-}uc(\alpha)}_{f_\mu})$, we have

$$\Re\left(1+\frac{zf_\mu''(z)}{f_\mu'(z)}\right)-\beta\left|\frac{zf_\mu''(z)}{f_\mu'(z)}\right|-\alpha>0$$

if and only if $r^{\beta\text{-}uc(\alpha)}_{f_\mu}$ is the unique root of

$$1+(1+\beta)\frac{rf_\mu''(r)}{f_\mu'(r)}=\alpha,\quad \alpha\in[0,1) \text{ and } \beta\geq 0.$$

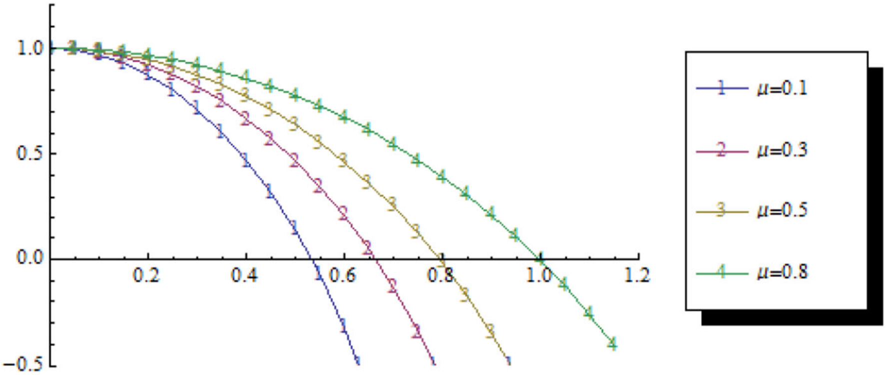
Thus the proof is complete. □

As a result of the Theorem 2.3, the following corollary is obtained by taking $\alpha = 0$ and $\beta = 1$.

Corollary 2.4 *Let* $\mu \in (0, 1)$. *Then, the radius of uniform convexity of the function* f_μ *is the smallest positive root of the equation*

$$1 + 2\left(\frac{rs''_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{s'_{\mu-\frac{1}{2},\frac{1}{2}}(r)} + \left(\frac{1}{\mu+\frac{1}{2}} - 1\right)\frac{rs'_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{s_{\mu-\frac{1}{2},\frac{1}{2}}(r)}\right) = 0.$$

Moreover, $r^{uc}_{f_\mu} < r^c_{f_\mu} < \xi'_{\mu,1} < \xi_{\mu,1}$, *where* $\xi_{\mu,1}$ *and* $\xi'_{\mu,1}$ *denote the first positive zeros of* $s_{\mu-\frac{1}{2},\frac{1}{2}}$ *and* $s'_{\mu-\frac{1}{2},\frac{1}{2}}$, *respectively, and* $r^c_{f_\mu}$ *is the radius of convexity of the function* f_μ.



The graph of the function $r \mapsto 1 + 2\left(\frac{rs''_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{s'_{\mu-\frac{1}{2},\frac{1}{2}}(r)} + \left(\frac{1}{\mu+\frac{1}{2}} - 1\right)\frac{rs'_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{s_{\mu-\frac{1}{2},\frac{1}{2}}(r)}\right)$ for $\mu \in \{0.1, 0.3, 0.5, 0.8\}$ on $[0, 1.2]$

Theorem 2.5 *Let* $\mu \in (-1, 1)$, $\mu \neq 0$, $\beta \geq 0$ *and* $\alpha \in [0, 1)$. *Then, the radius of* β*-uniform convexity of order* α *of the function* g_μ *is the smallest positive root of the equation*

$$(1-\alpha) - (1+\beta)\left(\frac{1}{2} + \mu - r\frac{(\frac{3}{2}-\mu)s'_{\mu-\frac{1}{2},\frac{1}{2}}(r) + rs''_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{(\frac{1}{2}-\mu)s_{\mu-\frac{1}{2},\frac{1}{2}}(r) + rs'_{\mu-\frac{1}{2},\frac{1}{2}}(r)}\right) = 0.$$

Moreover, $r^{\beta\text{-}uc(\alpha)}_{g_\mu} < r^c_{g_\mu} < \gamma'_{\mu,1} < \xi_{\mu,1}$, *where* $\xi_{\mu,1}$ *and* $\gamma_{\mu,1}$ *denote the first positive zeros of* $s_{\mu-\frac{1}{2},\frac{1}{2}}$ *and* g'_μ, *respectively.*

Proof Let $\xi_{\mu,n}$ and $\gamma_{\mu,n}$ denote the n-th positive root of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and g'_μ, respectively, and the smallest positive root of g'_μ does not exceed the first positive root of $s_{\mu-\frac{1}{2},\frac{1}{2}}$.

In [10] with the help of Hadamard's Theorem [18, p. 26], the following equality was proved:

$$1+\frac{zg_\mu''(z)}{g_\mu'(z)}=1-\sum_{n\geq 1}\frac{2z^2}{\gamma_{\mu,n}^2-z^2}.$$

By using inequality (1.5), for all $z\in U(\gamma_{\mu,1})$, we have the inequality

$$\Re\left(1+\frac{zg_\mu''(z)}{g_\mu'(z)}\right)\geq 1-\sum_{n\geq 1}\frac{2r^2}{\gamma_{\mu,n}^2-r^2}, \tag{2.8}$$

where $|z|=r$.

On the other hand, again by using inequality (1.5), for all $z\in U(\gamma_{\mu,1})$ and $\beta\geq 0$, we get the inequality

$$\begin{aligned}\beta\left|\frac{zg_\mu''(z)}{g_\mu'(z)}\right|&=\beta\left|\sum_{n\geq 1}\frac{2z^2}{\gamma_{\mu,n}^2-z^2}\right|\\&\leq\beta\sum_{n\geq 1}\left|\frac{2z^2}{\gamma_{\mu,n}^2-z^2}\right|\\&\leq\beta\sum_{n\geq 1}\frac{2r^2}{\gamma_{\mu,n}^2-r^2}=-\beta\frac{rg_\mu''(r)}{g_\mu'(r)}.\end{aligned} \tag{2.9}$$

Finally, the following inequality is inferred from (2.8) and (2.9):

$$\Re\left(1+\frac{zg_\mu''(z)}{g_\mu'(z)}\right)-\beta\left|\frac{zg_\mu''(z)}{g_\mu'(z)}\right|-\alpha\geq 1-\alpha+(1+\beta)\frac{rg_\mu''(r)}{g_\mu'(r)},\quad \beta\geq 0,\alpha\in[0,1),$$

where $|z|=r$. Thus, for $r\in(0,\gamma_{\mu,1})$, $\beta\geq 0$ and $\alpha\in[0,1)$, we obtain

$$\inf_{|z|<r}\left[\Re\left(1+\frac{zg_\mu''(z)}{g_\mu'(z)}\right)-\beta\left|\frac{zg_\mu''(z)}{g_\mu'(z)}\right|-\alpha\right]=1-\alpha+(1+\beta)\frac{rg_\mu''(r)}{g_\mu'(r)}.$$

The mapping $\Theta_\mu:(0,\gamma_{\mu,1})\to\mathbb{R}$ defined by

$$\Theta_\mu(r)=1-\alpha+(1+\beta)\frac{rg_\mu''(r)}{g_\mu'(r)}=1-\alpha-(1+\beta)\sum_{n\geq 1}\frac{2r^2}{\gamma_{\mu,n}^2-r^2}$$

is strictly decreasing since $\lim_{r\searrow 0}\Theta_\mu(r)=1$ and $\lim_{r\nearrow\gamma_{\mu,1}}\Theta_\mu(r)=-\infty$. As a result, in view of the minimum principle for harmonic functions it follows that for $\alpha\in[0,1)$, $\beta\geq 0$ and $z\in U(r_1)$, we have

$$\Re\left(1+\frac{zg''_\mu(z)}{g'_\mu(z)}\right)-\beta\left|\frac{zg''_\mu(z)}{g'_\mu(z)}\right|-\alpha>0$$

if and only if r_1 is the unique root of

$$1+(1+\beta)\frac{rg''_\mu(r)}{g'_\mu(r)}=\alpha,\quad \beta\geq 0 \text{ and } \alpha\in[0,1)$$

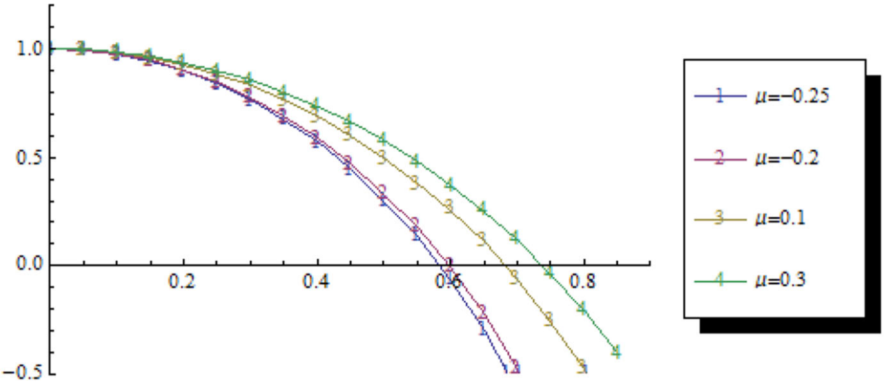
situated in $(0,\gamma_{\mu,1})$. □

As a result of the Theorem 2.5, the next corollary is obtained by taking $\alpha=0$ and $\beta=1$.

Corollary 2.6 *Let* $\mu\in(-1,1)$ *and* $\mu\neq 0$. *Then, the radius of uniform convexity of the function* g_μ *is the smallest positive root of the equation*

$$1-2\left(\frac{1}{2}+\mu-r\frac{(\frac{3}{2}-\mu)s'_{\mu-\frac{1}{2},\frac{1}{2}}(r)+rs''_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{(\frac{1}{2}-\mu)s_{\mu-\frac{1}{2},\frac{1}{2}}(r)+rs'_{\mu-\frac{1}{2},\frac{1}{2}}(r)}\right)=0.$$

Moreover, $r^{uc}_{g_\mu}<r^{c}_{g_\mu}<\gamma'_{\mu,1}<\xi_{\mu,1}$, *where* $\xi_{\mu,1}$ *and* $\gamma_{\mu,1}$ *denote the first positive zeros of* $s_{\mu-\frac{1}{2},\frac{1}{2}}$ *and* g'_μ, *respectively.*



The graph of the function $r\mapsto 1-2\left(\frac{1}{2}+\mu-r\frac{(\frac{3}{2}-\mu)s'_{\mu-\frac{1}{2},\frac{1}{2}}(r)+rs''_{\mu-\frac{1}{2},\frac{1}{2}}(r)}{(\frac{1}{2}-\mu)s_{\mu-\frac{1}{2},\frac{1}{2}}(r)+rs'_{\mu-\frac{1}{2},\frac{1}{2}}(r)}\right)$ for $\mu\in\{-0.25,-0.2,0.1,0.3\}$ on $[0,0.9]$

Theorem 2.7 *Let* $\mu\in(-1,1)$, $\mu\neq 0$, $\beta\geq 0$ *and* $\alpha\in[0,1)$. *Then, the radius of* β*-uniform convexity of order* α *of the function* h_μ *is the smallest positive root of the equation*

$$4(1-\alpha)-(1+\beta)\left(1+2\mu-2\sqrt{r}\frac{(\frac{5}{2}-\mu)s'_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})+\sqrt{r}s''_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})}{(\frac{3}{2}-\mu)s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})+\sqrt{r}s'_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})}\right)=0.$$

Moreover, $r_{h_\mu}^{\beta\text{-}uc(\alpha)} < r_{h_\mu}^{c} < \delta'_{\mu,1} < \xi_{\mu,1}$, *where* $\xi_{\mu,1}$ *and* $\delta_{\mu,1}$ *denote the first positive zeros of* $s_{\mu-\frac{1}{2},\frac{1}{2}}$ *and* h'_μ*, respectively.*

Proof Let $\xi_{\mu,n}$ and $\delta_{\mu,n}$ denote the n-th positive root of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and h'_μ, respectively, and the smallest positive root of h'_μ does not exceed the first positive root of $s_{\mu-\frac{1}{2},\frac{1}{2}}$. In [10] with the help of Hadamard's Theorem [18, p. 26], the following equality was proved:

$$1+\frac{zh''_\mu(z)}{h'_\mu(z)}=1-\sum_{n\geq 1}\frac{z}{\delta_{\mu,n}^2-z}.$$

By using inequality (1.5), for all $z\in U(\delta_{\mu,1})$ we obtain the inequality

$$\Re\left(1+\frac{zh''_\mu(z)}{h'_\mu(z)}\right)\geq 1-\sum_{n\geq 1}\frac{r}{\delta_{\mu,n}^2-r}, \tag{2.10}$$

where $|z|=r$.

Moreover, again by using inequality (1.5), for all $z\in U(\delta_{\mu,1})$ and $\beta\geq 0$, we get the inequality

$$\begin{aligned}\beta\left|\frac{zh''_\mu(z)}{h'_\mu(z)}\right| &= \beta\left|\sum_{n\geq 1}\frac{z}{\delta_{\mu,n}^2-z}\right|\\ &\leq \beta\sum_{n\geq 1}\left|\frac{z}{\delta_{\mu,n}^2-z}\right|\\ &\leq \beta\sum_{n\geq 1}\frac{r}{\delta_{\mu,n}^2-r}=-\beta\frac{rh''_\mu(r)}{h'_\mu(r)}. \end{aligned} \tag{2.11}$$

As a result, from (2.10) and (2.11), we have

$$\Re\left(1+\frac{zh''_\mu(z)}{h'_\mu(z)}\right)-\beta\left|\frac{zh''_\mu(z)}{h'_\mu(z)}\right|-\alpha\geq 1-\alpha+(1+\beta)\frac{rh''_\mu(r)}{h'_\mu(r)},$$

where $|z|=r$. Thus, for $r\in(0,\delta_{\mu,1})$, $\beta\geq 0$ and $\alpha\in[0,1)$, we have

$$\inf_{|z|<r}\left[\Re\left(1+\frac{zh''_\mu(z)}{h'_\mu(z)}\right)-\beta\left|\frac{zh''_\mu(z)}{h'_\mu(z)}\right|-\alpha\right]=1-\alpha+(1+\beta)\frac{rh''_\mu(r)}{h'_\mu(r)}.$$

The mapping $\Phi_\mu : (0, \delta_{\mu,1}) \to \mathbb{R}$ defined by

$$\Phi_\mu(r) = 1 - \alpha + (1+\beta)\frac{rh''_\mu(r)}{h'_\mu(r)} = 1 - \alpha - (1+\beta)\sum_{n\geq 1}\frac{r}{\delta^2_{\mu,n} - r}$$

is strictly decreasing since $\lim_{r\searrow 0}\Phi_\mu(r) = 1 > \alpha$ and $\lim_{r\nearrow\delta_{\mu,1}}\Phi_\mu(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions it follows that for $\alpha \in [0, 1)$, $\beta \geq 0$ and $z \in U(r_2)$, we have

$$\Re\left(1 + \frac{zh''_\mu(z)}{h'_\mu(z)}\right) - \beta\left|\frac{zh''_\mu(z)}{h'_\mu(z)}\right| - \alpha > 0$$

if and only if r_2 is the unique root of

$$1 + (1+\beta)\frac{rh''_\mu(r)}{h'_\mu(r)} = \alpha, \quad \alpha \in [0, 1) \text{ and } \beta \geq 0$$

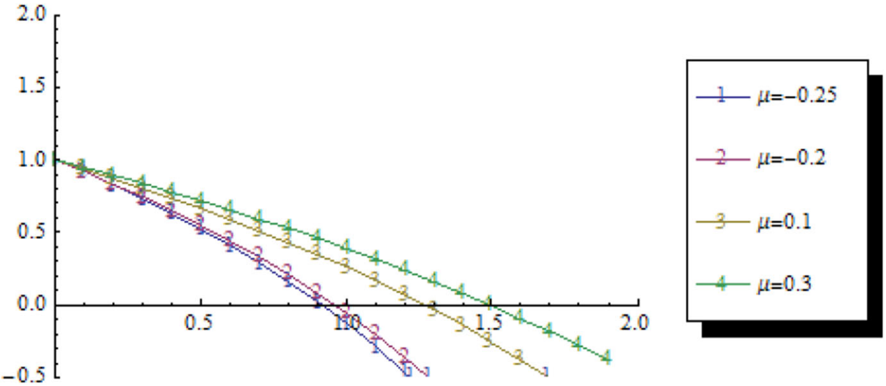
situated in $(0, \delta_{\mu,1})$. □

As a result of the Theorem 2.7, the following corollary is obtained by taking $\alpha = 0$ and $\beta = 1$.

Corollary 2.8 *Let* $\mu \in (-1, 1)$ *and* $\mu \neq 0$. *Then, the radius of uniform convexity of the function* h_μ *is the smallest positive root of the equation*

$$\frac{1}{2} - \mu + \sqrt{r}\,\frac{(\frac{5}{2} - \mu)s'_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r}) + \sqrt{r}s''_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})}{(\frac{3}{2} - \mu)s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r}) + \sqrt{r}s'_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})} = 0.$$

Moreover, $r^{uc}_{h_\mu} < r^{c}_{h_\mu} < \delta'_{\mu,1} < \xi_{\mu,1}$, *where* $\xi_{\mu,1}$ *and* $\delta_{\mu,1}$ *denote the first positive zeros of* $s_{\mu-\frac{1}{2},\frac{1}{2}}$ *and* h'_μ, *respectively.*



The graph of the function $r \mapsto \frac{1}{2} - \mu + \sqrt{r}\frac{(\frac{5}{2}-\mu)s'_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})+\sqrt{r}s''_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})}{(\frac{3}{2}-\mu)s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})+\sqrt{r}s'_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{r})}$ for $\mu \in \{-0.25, -0.2, 0.1, 0.3\}$ on $[0, 2]$

For $\mu = \frac{1}{3}$, Lommel functions defined in terms of the hypergeometric function ${}_1F_2$ as follows:

$$s_{-\frac{1}{5},\frac{1}{2}}(z) = \frac{100z^{4/5}}{39}\ {}_1F_2\left(1; \frac{23}{20}, \frac{33}{20}; -\frac{z^2}{4}\right).$$

Then, we have

$$f_{\frac{3}{10}}(z) = z\left[{}_1F_2\left(1; \frac{23}{20}, \frac{33}{20}; -\frac{z^2}{4}\right)\right]^{5/4}, \quad g_{\frac{3}{10}}(z) = z\ {}_1F_2\left(1; \frac{23}{20}, \frac{33}{20}; -\frac{z^2}{4}\right)$$

and

$$h_{\frac{3}{10}}(z) = z\ {}_1F_2\left(1; \frac{23}{20}, \frac{33}{20}; -\frac{z}{4}\right).$$

We obtain the following results for the functions $f_{\frac{3}{10}}$, $g_{\frac{3}{10}}$ and $h_{\frac{3}{10}}$:

- $f_{\frac{3}{10}}(z) \in UC$ in the disk $U(r_1 = 0.6623)$,
- $g_{\frac{3}{10}}(z) \in UC$ in the disk $U(r_2 = 0.7376)$,
- $h_{\frac{3}{10}}(z) \in UC$ in the disk $U(r_3 = 1.4961)$,

where r_1, r_2 and r_3 is the smallest positive root of the equations given Corollary 2.4, 2.6 and 2.8 for $\mu = \frac{1}{3}$.

Second, the other main result of this section presents the β-uniform convexity of order α of functions u_ν, v_ν and w_ν, related to Struve's one. The first part of next theorem is an interesting of Lemma 2.2.

Theorem 2.9 *Let $|\nu| \leq \frac{1}{2}$, $\beta \geq 0$ and $0 \leq \alpha < 1$. Then, the radius of β-uniform convexity of order α of the function u_ν is the smallest positive root of the equation*

$$(1-\alpha) + (1+\beta)\left(\frac{r\mathbf{H}''_\nu(r)}{\mathbf{H}'_\nu(r)} + \left(\frac{1}{\nu+1} - 1\right)\frac{r\mathbf{H}'_\nu(r)}{\mathbf{H}_\nu(r)}\right) = 0.$$

Moreover, $r_{u_\nu}^{\beta\text{-}uc(\alpha)} < r_{u_\nu}^{c} < h'_{\nu,1} < h_{\nu,1}$, where $h_{\nu,1}$ and $h'_{\nu,1}$ denote the first positive zeros of $\mathbf{H}_\nu$ and $\mathbf{H}'_\nu$, respectively.

Proof We note that

$$1 + \frac{zu''_\nu(z)}{u'_\nu(z)} = 1 + \frac{z\mathbf{H}''_\nu(z)}{\mathbf{H}'_\nu(z)} + \left(\frac{1}{\nu+1} - 1\right)\frac{z\mathbf{H}'_\nu(z)}{\mathbf{H}_\nu(z)}.$$

Using the Mittag–Leffler expansions of $\mathbf{H}_\nu$ and $\mathbf{H}'_\nu$ [10, Theorem 4] given by

$$\mathbf{H}_\nu(z) = \frac{z^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\prod_{n\geq 1}\left(1 - \frac{z^2}{h^2_{\nu,n}}\right) \tag{2.12}$$

and

$$\mathbf{H}'_\nu(z) = \frac{(\nu+1)\,z^\nu}{\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)} \prod_{n\geq1}\left(1-\frac{z^2}{h'^2_{\nu,n}}\right) \tag{2.13}$$

where $h_{\nu,n}$ and $h'_{\nu,n}$ denote the n-th positive root of $\mathbf{H}_\nu$ and $\mathbf{H}'_\nu$, respectively. From (2.12) and (2.13), we obtain

$$\frac{z\mathbf{H}'_\nu(z)}{\mathbf{H}_\nu(z)} = \nu+1-\sum_{n\geq1}\frac{2z^2}{h^2_{\nu,n}-z^2}, \quad 1+\frac{z\mathbf{H}''_\nu(z)}{\mathbf{H}'_\nu(z)} = \nu+1-\sum_{n\geq1}\frac{2z^2}{h'^2_{\nu,n}-z^2}.$$

Thus, we have

$$1+\frac{zu''_\nu(z)}{u'_\nu(z)} = 1-\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq1}\frac{2z^2}{h^2_{\nu,n}-z^2}-\sum_{n\geq1}\frac{2z^2}{h'^2_{\nu,n}-z^2}.$$

Now, the proof will be presented in two cases by considering the intervals of ν. First, suppose that $\nu\in\left[-\frac{1}{2},0\right]$. Since $\frac{1}{\nu+1}-1\geq0$, inequality (1.5) implies

$$\begin{aligned}\Re\left(1+\frac{zu''_\nu(z)}{u'_\nu(z)}\right) &= 1-\sum_{n\geq1}\Re\left(\frac{2z^2}{h'^2_{\nu,n}-z^2}\right)-\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq1}\Re\left(\frac{2z^2}{h^2_{\nu,n}-z^2}\right)\\ &\geq 1-\sum_{n\geq1}\frac{2r^2}{h'^2_{\nu,n}-r^2}-\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq1}\frac{2r^2}{h^2_{\nu,n}-r^2}\\ &= 1+\frac{ru''_\nu(r)}{u'_\nu(r)}.\end{aligned} \tag{2.14}$$

On the other hand, if in the second part of inequality (1.5) we replace z by z^2 and b by $h'_{\nu,1}$ and $h_{\nu,1}$, respectively, then it follows that

$$\left|\frac{2z^2}{h'^2_{\nu,n}-z^2}\right| \leq \frac{2r^2}{h'^2_{\nu,n}-r^2} \text{ and } \left|\frac{2z^2}{h^2_{\nu,n}-z^2}\right| \leq \frac{2r^2}{h^2_{\nu,n}-r^2},$$

provided that $|z|\leq r<h'_{\nu,1}<h_{\nu,1}$. These two inequalities and the conditions $\frac{1}{\nu+1}-1\geq0$ and $\beta\geq0$ imply that

$$\begin{aligned}\beta\left|\frac{zu''_\nu(z)}{u'_\nu(z)}\right| &= \beta\left|\sum_{n\geq1}\left(\frac{2z^2}{h'^2_{\nu,n}-z^2}+\left(\frac{1}{\nu+1}-1\right)\frac{2z^2}{h^2_{\nu,n}-z^2}\right)\right|\\ &\leq \beta\sum_{n\geq1}\left|\frac{2z^2}{h'^2_{\nu,n}-z^2}\right|+\beta\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq1}\left|\frac{2z^2}{h^2_{\nu,n}-z^2}\right|\end{aligned}$$

$$\leq \beta \sum_{n\geq 1} \left(\frac{2r^2}{h'^2_{\nu,n} - r^2} + \left(\frac{1}{\nu+1} - 1 \right) \frac{2r^2}{h^2_{\nu,n} - r^2} \right) = -\beta \frac{r u''_\nu(r)}{u'_\nu(r)}. \tag{2.15}$$

From (2.14) and (2.15), we get

$$\Re \left(1 + \frac{z u''_\nu(z)}{u'_\nu(z)} \right) - \beta \left| \frac{z u''_\nu(z)}{u'_\nu(z)} \right| - \alpha \geq 1 - \alpha + (1+\beta) \frac{r u''_\nu(r)}{u'_\nu(r)},$$

where $|z| \leq r < h'_{\nu,1}$, $\beta \geq 0$ and $\alpha \in [0, 1)$.

Second, in the case $\nu \in \left[0, \frac{1}{2}\right]$, the roots $0 < h'_{\nu,1} < h_{\nu,1}$ are real for every natural number n. Moreover, inequality (1.5) implies that

$$\Re \left(\frac{2z^2}{h'^2_{\nu,n} - z^2} \right) \leq \left| \frac{2z^2}{h'^2_{\nu,n} - z^2} \right| \leq \frac{2r^2}{h'^2_{\nu,n} - r^2}, \quad |z| \leq r < h'_{\nu,1} < h_{\nu,1}$$

and

$$\Re \left(\frac{2z^2}{h^2_{\nu,n} - z^2} \right) \leq \left| \frac{2z^2}{h^2_{\nu,n} - z^2} \right| \leq \frac{2r^2}{h^2_{\nu,n} - r^2}, \quad |z| \leq r < h'_{\nu,1} < h_{\nu,1}.$$

Putting $\lambda = 1 - \frac{1}{\nu+1}$, inequality (1.4) implies

$$\Re \left(\frac{2z^2}{h'^2_{\nu,n} - z^2} - \left(1 - \frac{1}{\nu+1} \right) \frac{2z^2}{h^2_{\nu,n} - z^2} \right) \leq \frac{2r^2}{h'^2_{\nu,n} - r^2} - \left(1 - \frac{1}{\nu+1} \right) \frac{2r^2}{h^2_{\nu,n} - r^2}$$

for $|z| \leq r < h'_{\nu,1} < h_{\nu,1}$, and we get

$$\begin{aligned} \Re \left(1 + \frac{z u''_\nu(z)}{u'_\nu(z)} \right) &= 1 - \sum_{n\geq 1} \Re \left(\frac{2z^2}{h'^2_{\nu,n} - z^2} - \left(1 - \frac{1}{\nu+1} \right) \frac{2z^2}{h^2_{\nu,n} - z^2} \right) \\ &\geq 1 - \sum_{n\geq 1} \left(\frac{2r^2}{h'^2_{\nu,n} - r^2} - \left(1 - \frac{1}{\nu+1} \right) \frac{2r^2}{h^2_{\nu,n} - r^2} \right) \\ &= 1 + \frac{r u''_\nu(r)}{u'_\nu(r)}. \end{aligned} \tag{2.16}$$

Now, if in the inequality (1.3) we replace z by z^2 and we again put $\lambda = 1 - \frac{1}{\nu+1}$, it follows that

$$\left| \frac{2z^2}{h'^2_{\nu,n} - z^2} - \left(1 - \frac{1}{\nu+1} \right) \frac{2z^2}{h^2_{\nu,n} - z^2} \right| \leq \frac{2r^2}{h'^2_{\nu,n} - r^2} - \left(1 - \frac{1}{\nu+1} \right) \frac{2r^2}{h^2_{\nu,n} - r^2},$$

provided that $|z| \leq r < h'_{\nu,1} < h_{\nu,1}$. Thus, for $\beta \geq 0$, we have

$$\begin{aligned} \beta \left| \frac{zu''_\nu(z)}{u'_\nu(z)} \right| &= \beta \left| \sum_{n\geq 1} \left(\frac{2z^2}{h'^2_{\nu,n} - z^2} - \left(1 - \frac{1}{\nu+1}\right) \frac{2z^2}{h^2_{\nu,n} - z^2} \right) \right| \\ &\leq \beta \sum_{n\geq 1} \left| \frac{2z^2}{h'^2_{\nu,n} - z^2} - \left(1 - \frac{1}{\nu+1}\right) \frac{2z^2}{h^2_{\nu,n} - z^2} \right| \\ &\leq \beta \sum_{n\geq 1} \left(\frac{2r^2}{h'^2_{\nu,n} - r^2} - \left(1 - \frac{1}{\nu+1}\right) \frac{2r^2}{h^2_{\nu,n} - r^2} \right) = -\beta \frac{ru''_\nu(r)}{u'_\nu(r)}. \end{aligned} \tag{2.17}$$

As a result, the following inequality be inferred from (2.16) and (2.17) such as (2.14) and (2.15),

$$\Re \left(1 + \frac{zu''_\nu(z)}{u'_\nu(z)} \right) - \beta \left| \frac{zu''_\nu(z)}{u'_\nu(z)} \right| - \alpha \geq 1 - \alpha + (1+\beta) \frac{ru''_\nu(r)}{u'_\nu(r)}, \tag{2.18}$$

where $|z| \leq r < h'_{\nu,1}$, $\beta \geq 0$ and $\alpha \in [0, 1)$.

Equality holds (2) and (2.18) if and only if $z = r$. Thus, it follows that

$$\inf_{|z|<r} \left[\Re \left(1 + \frac{zu''_\nu(z)}{u'_\nu(z)} \right) - \beta \left| \frac{zu''_\nu(z)}{u'_\nu(z)} \right| - \alpha \right] = 1 - \alpha + (1+\beta) \frac{ru''_\nu(r)}{u'_\nu(r)},$$

where $r \in (0, h'_{\nu,1})$, $\beta \geq 0$ and $\alpha \in [0, 1)$.

The mapping $\psi_\nu : (0, h'_{\nu,1}) \to \mathbb{R}$ defined by

$$\psi_\nu(r) = 1+(1+\beta) \frac{ru''_\nu(r)}{u'_\nu(r)} = 1-(1+\beta) \sum_{n\geq 1} \left(\frac{2r^2}{h'^2_{\nu,n} - r^2} - \left(1 - \frac{1}{\nu+1}\right) \frac{2r^2}{h^2_{\nu,n} - r^2} \right)$$

is strictly decreasing for all $|\nu| \leq \frac{1}{2}$ and $\beta \geq 0$. Namely, we obtain

$$\begin{aligned} \psi'_\nu(r) &= -(1+\beta) \sum_{n\geq 1} \left(\frac{4rh'^2_{\nu,n}}{\left(h'^2_{\nu,n} - r^2\right)^2} - \left(1 - \frac{1}{\nu+1}\right) \frac{4rh^2_{\nu,n}}{\left(h^2_{\nu,n} - r^2\right)^2} \right) \\ &< (1+\beta) \sum_{n\geq 1} \left(\frac{4rh^2_{\nu,n}}{\left(h^2_{\nu,n} - r^2\right)^2} - \frac{4rh'^2_{\nu,n}}{\left(h'^2_{\nu,n} - r^2\right)^2} \right) < 0 \end{aligned}$$

for $\nu \in \left[0, \frac{1}{2}\right]$, $r \in (0, h'_{\nu,1})$ and $\beta \geq 0$. Here, we used again that the zeros $h_{\nu,n}$ and $h'_{\nu,n}$ interlace, and for all $n \in \mathbb{N}$, $|\nu| \leq \frac{1}{2}$ and $r < \sqrt{h_{\nu,n} h'_{\nu,n}}$, we have that

$$h^2_{\nu,n} \left(h'^2_{\nu,n} - r^2\right)^2 < h'^2_{\nu,n} \left(h^2_{\nu,n} - r^2\right)^2.$$

Observe that when $\nu \in \left[\frac{1}{2}, 0\right]$ and $r > 0$ we have also that $\psi_\nu'(r) < 0$, and thus ψ_ν is indeed strictly decreasing for all $|\nu| \leq \frac{1}{2}$ and $\beta \geq 0$.

Now, since $\lim_{r\searrow 0} \psi_\nu(r) = 1$ and $\lim_{r\nearrow h'_{\nu,1}} \psi_\nu(r) = -\infty$, in view of the minimum principle for harmonic functions it follows that for $|\nu| \leq \frac{1}{2}$ and $z \in U(r_3)$, we get

$$\Re\left(1 + \frac{zu_\nu''(z)}{u_\nu'(z)}\right) - \beta\left|\frac{zu_\nu''(z)}{u_\nu'(z)}\right| > \alpha$$

if and only if r_3 is the unique root of

$$1 + (1+\beta)\frac{ru_\nu''(r)}{u_\nu'(r)} = \alpha, \quad \alpha \in [0, 1) \text{ and } \beta \geq 0$$

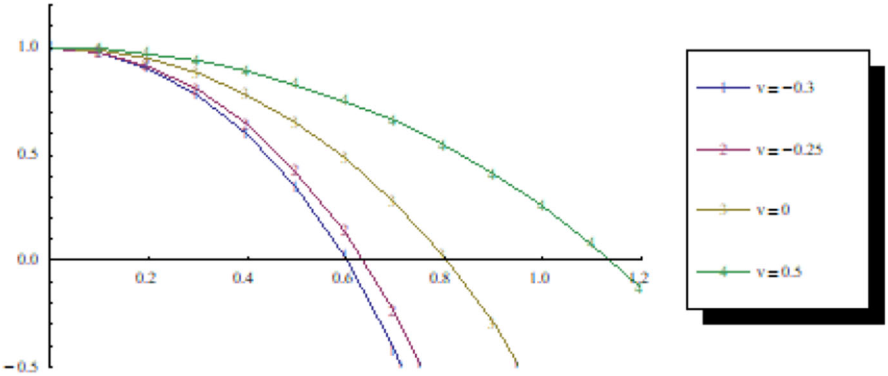
situated in $\left(0, h'_{\nu,1}\right)$. ☐

As a result of the Theorem 2.9, the next corollary is obtained by taking $\alpha = 0$ and $\beta = 1$.

Corollary 2.10 *Let $|\nu| \leq \frac{1}{2}$. Then, the radius of uniform convexity of the function u_ν is the smallest positive root of the equation*

$$1 + 2\left(\frac{r\mathbf{H}_\nu''(r)}{\mathbf{H}_\nu'(r)} + \left(\frac{1}{\nu+1} - 1\right)\frac{r\mathbf{H}_\nu'(r)}{\mathbf{H}_\nu(r)}\right) = 0.$$

Moreover, $r^{uc}_{u_\nu} < r^{c}_{u_\nu} < h'_{\nu,1} < h_{\nu,1}$, where $h_{\nu,1}$ and $h'_{\nu,1}$ denote the first positive zeros of $\mathbf{H}_\nu$ and $\mathbf{H}_\nu'$, respectively.



The graph of the function $r \mapsto 1 + 2\left(\frac{r\mathbf{H}_\nu''(r)}{\mathbf{H}_\nu'(r)} + \left(\frac{1}{\nu+1} - 1\right)\frac{r\mathbf{H}_\nu'(r)}{\mathbf{H}_\nu(r)}\right)$ for $\nu \in \{-0.3, -0.25, 0, 0.5\}$ on $[0, 1.2]$

Theorem 2.11 *Let $|\nu| \leq \frac{1}{2}$, $\beta \geq 0$ and $0 \leq \alpha < 1$. Then, the radius of β-uniform convexity of order α of the function v_ν is the smallest positive root of the equation*

$$(1-\alpha) - (1+\beta)\left(1 + \nu - r\frac{(1-\nu)\mathbf{H}_\nu'(r) + r\mathbf{H}_\nu''(r)}{-\nu\mathbf{H}_\nu(r) + r\mathbf{H}_\nu'(r)}\right) = 0.$$

Moreover, $r_{v_\nu}^{\beta\text{-}uc(\alpha)} < r_{v_\nu}^{c} < \varsigma_{\nu,1} < h_{\nu,1}$, *where* $h_{\nu,1}$ *and* $\varsigma_{\nu,1}$ *denote the first positive zeros of* $\mathbf{H}_\nu$ *and* v_ν', *respectively.*

Proof Let $h_{\nu,n}$ and $\varsigma_{\nu,n}$ denote the n-th positive root of $\mathbf{H}_\nu$ and v_ν', respectively, and the smallest positive root of v_ν' does not exceed the first positive root of $\mathbf{H}_\nu$. In [10], the following equality was proved:

$$1+\frac{zv_\nu''(z)}{v_\nu'(z)}=-\nu+z\frac{(1-\nu)\mathbf{H}_\nu'(z)+z\mathbf{H}_\nu''(z)}{-\nu\mathbf{H}_\nu(z)+z\mathbf{H}_\nu'(z)}=1-\sum_{n\geq 1}\frac{2z^2}{\varsigma_{\nu,n}^2-z^2}.$$

By using inequality (1.5), for all $z\in U(\varsigma_{\nu,1})$, we have the inequality

$$\Re\left(1+\frac{zv_\nu''(z)}{v_\nu'(z)}\right)\geq 1-\sum_{n\geq 1}\frac{2r^2}{\varsigma_{\nu,n}^2-r^2}, \tag{2.19}$$

where $|z|=r$.

On the other hand, again by using inequality (1.5), for all $z\in U(\varsigma_{\nu,1})$ and $\beta\geq 0$, we get the inequality

$$\begin{aligned}\beta\left|\frac{zv_\nu''(z)}{v_\nu'(z)}\right| &= \beta\left|\sum_{n\geq 1}\frac{2z^2}{\varsigma_{\nu,n}^2-z^2}\right|\\ &\leq \beta\sum_{n\geq 1}\left|\frac{2z^2}{\varsigma_{\nu,n}^2-z^2}\right|\\ &\leq \beta\sum_{n\geq 1}\frac{2r^2}{\varsigma_{\nu,n}^2-r^2}=-\beta\frac{rv_\nu''(r)}{v_\nu'(r)}. \end{aligned}\tag{2.20}$$

Finally, the following inequality is inferred from (2.19) and (2.20):

$$\Re\left(1+\frac{zv_\nu''(z)}{v_\nu'(z)}\right)-\beta\left|\frac{zv_\nu''(z)}{v_\nu'(z)}\right|-\alpha\geq 1-\alpha+(1+\beta)\frac{rv_\nu''(r)}{v_\nu'(r)},\quad \beta\geq 0,$$

where $|z|=r$. Thus, for $r\in(0,\varsigma_{\nu,1})$, $\beta\geq 0$ and $\alpha\in[0,1)$, we obtain

$$\inf_{|z|<r}\left[\Re\left(1+\frac{zv_\nu''(z)}{v_\nu'(z)}\right)-\beta\left|\frac{zv_\nu''(z)}{v_\nu'(z)}\right|-\alpha\right]=1-\alpha+(1+\beta)\frac{rv_\nu''(r)}{v_\nu'(r)}.$$

The mapping $\Theta_\nu:(0,\varsigma_{\nu,1})\to\mathbb{R}$ defined by

$$\Theta_\nu(r)=1+(1+\beta)\frac{rv_\nu''(r)}{v_\nu'(r)}=1-(1+\beta)\sum_{n\geq 1}\frac{2r^2}{\varsigma_{\nu,n}^2-r^2}$$

is strictly decreasing since $\lim_{r\searrow 0}\Theta_\nu(r) = 1$ and $\lim_{r\nearrow\varsigma_{\nu,1}}\Theta_\nu(r) = -\infty$. As a result, in view of the minimum principle for harmonic functions it follows that for $\alpha \in [0, 1)$, $\beta \geq 0$ and $z \in U(r_4)$, we have

$$\Re\left(1 + \frac{zv_\nu''(z)}{v_\nu'(z)}\right) - \beta\left|\frac{zv_\nu''(z)}{v_\nu'(z)}\right| > \alpha$$

if and only if r_4 is the unique root of

$$1 + (1+\beta)\frac{rv_\nu''(r)}{v_\nu'(r)} = \alpha, \quad \alpha \in [0, 1) \text{ and } \beta \geq 0$$

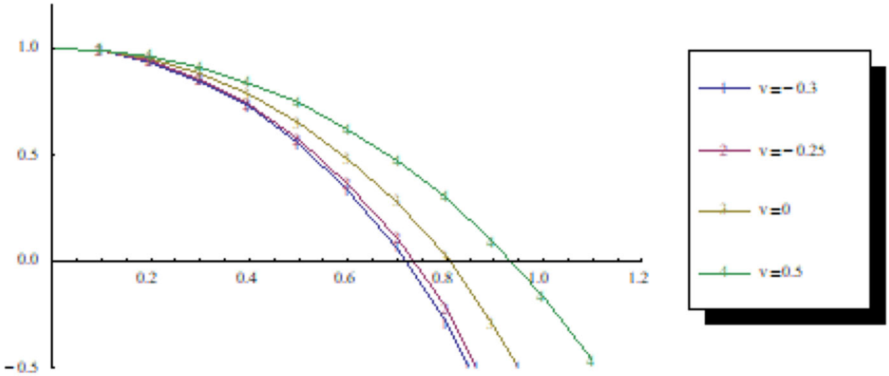
situated in $(0, \varsigma_{\nu,1})$. □

As a result of the Theorem 2.11, the following corollary is obtained by taking $\alpha = 0$ and $\beta = 1$:

Corollary 2.12 *Let $|\nu| \leq \frac{1}{2}$. Then, the radius of uniform convexity of the function v_ν is the smallest positive root of the equation*

$$1 - 2\left(1 + \nu - r\frac{(1-\nu)\mathbf{H}_\nu'(r) + r\mathbf{H}_\nu''(r)}{-\nu\mathbf{H}_\nu(r) + r\mathbf{H}_\nu'(r)}\right) = 0.$$

Moreover, $r_{v_\nu}^{uc} < r_{v_\nu}^{c} < \varsigma_{\nu,1} < h_{\nu,1}$, where $h_{\nu,1}$ and $\varsigma_{\nu,1}$ denote the first positive zeros of $\mathbf{H}_\nu$ and v_ν', respectively.



The graph of the function $r \mapsto 1 - 2\left(1 + \nu - r\frac{(1-\nu)\mathbf{H}_\nu'(r)+r\mathbf{H}_\nu''(r)}{-\nu\mathbf{H}_\nu(r)+r\mathbf{H}_\nu'(r)}\right)$ for $\nu \in \{-0.3, -0.25, 0, 0.5\}$ on $[0, 1.2]$

Theorem 2.13 *Let $|\nu| \leq \frac{1}{2}$, $\beta \geq 0$ and $0 \leq \alpha < 1$. Then, the radius of β-uniformly convex of order α of the function w_ν is the smallest positive root of the equation*

$$2(1-\alpha) - (1+\beta)\left(1 + \nu - \sqrt{r}\frac{(2-\nu)\mathbf{H}_\nu'(\sqrt{r}) + \sqrt{r}\mathbf{H}_\nu''(\sqrt{r})}{(1-\nu)\mathbf{H}_\nu(\sqrt{r}) + \sqrt{r}\mathbf{H}_\nu'(\sqrt{r})}\right) = 0.$$

Moreover, $r_{w_\nu}^{\beta\text{-}uc(\alpha)} < r_{w_\nu}^{c} < \sigma_{\nu,1} < h_{\nu,1}$, *where* $h_{\nu,1}$ *and* $\sigma_{\nu,1}$ *denote the first positive zeros of* $\mathbf{H}_\nu$ *and* w_ν', *respectively.*

Proof Let $h_{\nu,n}$ and $\sigma_{\nu,n}$ denote the n-th positive root of $\mathbf{H}_\nu$ and w_ν', respectively, and the smallest positive root of w_ν' does not exceed the first positive root of $\mathbf{H}_\nu$. In [10], the following equality was proved:

$$1+\frac{zw_\nu''(z)}{w_\nu'(z)}=\frac{1}{2}\left[1-\nu+\sqrt{r}\frac{(2-\nu)\mathbf{H}_\nu'(\sqrt{r})+\sqrt{r}\mathbf{H}_\nu''(\sqrt{r})}{(1-\nu)\mathbf{H}_\nu(\sqrt{r})+\sqrt{r}\mathbf{H}_\nu'(\sqrt{r})}\right]=1-\sum_{n\geq 1}\frac{z}{\sigma_{\nu,n}^2-z}.$$

By using inequality (1.5), for all $z\in U(\sigma_{\nu,1})$, we obtain the inequality

$$\Re\left(1+\frac{zw_\nu''(z)}{w_\nu'(z)}\right)\geq 1-\sum_{n\geq 1}\frac{r}{\sigma_{\nu,n}^2-r}, \tag{2.21}$$

where $|z|=r$.

Moreover, again by using inequality (1.5), for all $z\in U(\sigma_{\nu,1})$ and $\beta\geq 0$, we get the inequality

$$\begin{aligned}\beta\left|\frac{zw_\nu''(z)}{w_\nu'(z)}\right|&=\beta\left|\sum_{n\geq 1}\frac{z}{\sigma_{\nu,n}^2-z}\right|\\&\leq\beta\sum_{n\geq 1}\left|\frac{z}{\sigma_{\nu,n}^2-z}\right|\\&\leq\beta\sum_{n\geq 1}\frac{r}{\sigma_{\nu,n}^2-r}=-\beta\frac{rw_\nu''(r)}{w_\nu'(r)}.\end{aligned} \tag{2.22}$$

As a result, the following inequality is inferred from (2.21) and (2.22):

$$\Re\left(1+\frac{zw_\nu''(z)}{w_\nu'(z)}\right)-\beta\left|\frac{zw_\nu''(z)}{w_\nu'(z)}\right|-\alpha\geq 1-\alpha+(1+\beta)\frac{rw_\nu''(r)}{w_\nu'(r)},\ \ \beta\geq 0 \text{ and } \alpha\in[0,1),$$

where $|z|=r$. So, for $r\in(0,\sigma_{\nu,1})$, $\beta\geq 0$ and $\alpha\in[0,1)$, we have

$$\inf_{|z|<r}\left[\Re\left(1+\frac{zw_\nu''(z)}{w_\nu'(z)}\right)-\beta\left|\frac{zw_\nu''(z)}{w_\nu'(z)}\right|-\alpha\right]=1-\alpha+(1+\beta)\frac{rw_\nu''(r)}{w_\nu'(r)}.$$

The mapping $\Phi_\nu:(0,\sigma_{\nu,1})\to\mathbb{R}$ defined by

$$\Phi_\nu(r)=1+(1+\beta)\frac{rw_\nu''(r)}{w_\nu'(r)}=1-(1+\beta)\sum_{n\geq 1}\frac{r}{\sigma_{\nu,n}^2-r}$$

is strictly decreasing since $\lim_{r\searrow 0}\Phi_\nu(r) = 1 > \alpha$ and $\lim_{r\nearrow\sigma_{\nu,1}}\Phi_\nu(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions it follows that for $\alpha \in [0,1)$, $\beta \geq 0$ and $z \in U(r_5)$, we obtain

$$\Re\left(1+\frac{zw_\nu''(z)}{w_\nu'(z)}\right)-\beta\left|\frac{zw_\nu''(z)}{w_\nu'(z)}\right| > \alpha$$

if and only if r_5 is the unique root of

$$1+(1+\beta)\frac{rw_\nu''(r)}{w_\nu'(r)} = \alpha, \quad \alpha \in [0,1) \text{ and } \beta \geq 0$$

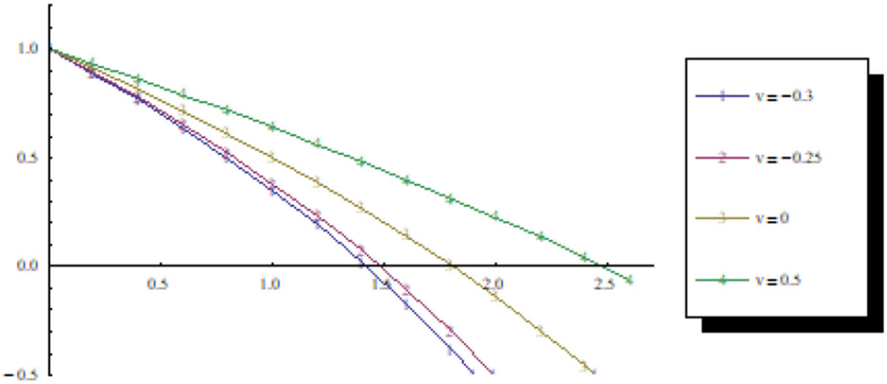
situated in $(0, \sigma_{\nu,1})$. ☐

As a result of the Theorem 2.13, the next corollary is obtained by taking $\alpha = 0$ and $\beta = 1$.

Corollary 2.14 *Let $|\nu| \leq \frac{1}{2}$. Then, the uniform convexity of the function w_ν is the smallest positive root of the equation*

$$-\nu+\sqrt{r}\frac{(2-\nu)\mathbf{H}_\nu'(\sqrt{r})+\sqrt{r}\mathbf{H}_\nu''(\sqrt{r})}{(1-\nu)\mathbf{H}_\nu(\sqrt{r})+\sqrt{r}\mathbf{H}_\nu'(\sqrt{r})} = 0.$$

Moreover, $r^{uc}_{w_\nu} < r^{c}_{w_\nu} < \sigma_{\nu,1} < h_{\nu,1}$, where $h_{\nu,1}$ and $\sigma_{\nu,1}$ denote the first positive zeros of $\mathbf{H}_\nu$ and w_ν', respectively.



The graph of the function $r \mapsto -\nu+\sqrt{r}\frac{(2-\nu)\mathbf{H}_\nu'(\sqrt{r})+\sqrt{r}\mathbf{H}_\nu''(\sqrt{r})}{(1-\nu)\mathbf{H}_\nu(\sqrt{r})+\sqrt{r}\mathbf{H}_\nu'(\sqrt{r})}$ for $\nu \in \{-0.3, -0.25, 0, 0.5\}$ on $[0, 2.7]$

Using the following representation of Struve functions of order $1/2$ in terms of elementary trigonometric functions

$$\mathbf{H}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left(1-\cos z\right),$$

we obtain

$$u_{\frac{1}{2}}(z) = 2^{\frac{2}{3}}\left(\frac{1-\cos z}{\sqrt{z}}\right)^{\frac{2}{3}}, \quad v_{\frac{1}{2}}(z) = 2\left(\frac{1-\cos z}{z}\right) \text{ and } w_{\frac{1}{2}}(z) = 2\left(1-\cos\sqrt{z}\right).$$

We state the following results for the functions: $u_{\frac{1}{2}},\ v_{\frac{1}{2}}$ and $w_{\frac{1}{2}}$:

- $u_{\frac{1}{2}}(z) \in UC$ in the disk $U(r_1 = 1.1382)$,
- $v_{\frac{1}{2}}(z) \in UC$ in the disk $U(r_2 = 0.9349)$,
- $w_{\frac{1}{2}}(z) \in UC$ in the disk $U(r_3 = 2.4674)$,

where $r_1,\ r_2$ and r_3 is the smallest positive root of the equations

- $5 + \left(-5 + 8r^2\right)\cos r - 2r\,(2r + \sin r) = 0$,
- $\cos r\,\left(2r^2 - 3\right) - r\sin r + 3 = 0$,
- $\sqrt{r}\cot\sqrt{r} = 0$, respectively.

Remark 2.15 For $\beta = 0$, Theorems 2.3, 2.5 and 2.7 reduce to [10, Thm. 3, (a),(b) and (c)], respectively.
Moreover, Theorems 2.9, 2.11 and 2.13 reduce to [10, Theorem 4, (a), (b) and (c)], respectively, by putting $\beta = 0$, for the case $|\nu| \le \frac{1}{2}$.

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