Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 5, No. 1, (January 2013), pp. 73–85
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Inverse and direct systems of soft modules

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Received 12 November 2011; Revised 21 May 2012; Accepted 22 May 2012

Abstract. Molodtsov initiated the concept of soft sets in [8]. Maji et al. defined some operations on soft sets in [22]. In this paper, we introduce the concept of inverse and direct systems in the category of soft modules. Finally, we investigate whether or not the limit of inverse system of soft modules.

2010 AMS Classification: 18A30

Keywords: Soft set, Soft module, Inverse system, Inverse limit.

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1. Introduction

To solve complicated problems in social sciences, economics, engineering and environment etc., we cannot use classical methods. The solutions of such problems involve the use of mathematical principles based and uncertainty and imprecision. Thus classical set theory, which is based on the crispand exact case may not be fully suitable for handling such problems of uncertainty. A number of theories have been propesed for dealing with unceartainties in an efficient way. Some of these are theory of fuzzy sets [3], theory of intuitionistic fuzzy sets [6], theory of vague sets, theory of interval mathematics [9][22] and theory of rough sets [20]. However, these theories have their own difficulties. In 1999, Molodtsov [8] introduced the concept of soft set theory which is a completely new approach for modeling uncertainty. He presented the fundamental results of the new theory and succesfully applied it to several directions such as smoothness of functions, game theory, operations research, Rieman-integration, Peron integration, theory of probability etc. Maji et al. [22][21] worked on soft set theory and presented an aplication of soft sets in decision making problems.

The theory of fuzzy sets,first developed by Zadeh [14] in 1965, perhaps the most appropriate framework for dealing with uncertainties. A fuzzy set is defined by its *membership function*, whose values are defined on the closed interval $[0, 1]$.

Some or all elements of a fuzzy set are thus assigned only partial membership. This approach puts the generalized theory of uncertainty into a much broader perspective, as outlined in [15]. The basic ideas of fuzzy set theory and its extensions, as well as many interesting applications, can be found in a number of books (e.g.,[7][10][13]).

H. Aktaş and N. Çağman [12] has established a connection betwen soft sets and fuzzy sets and they introduced soft groups.U. Acar and F. Koyuncu [24] introduced soft rings. Qiu-Mei Sun and his friends introduced soft modules[23].

L. Jin-Liang and A. Aygunoğlu [16][1] presented fuzzy soft sets and fuzzy soft groups. Ç. Gunduz and S. Bayramov [4] presented fuzzy soft modules.

The problem which obtained in new categories are closed according to algebraic operations is very important. Since inverse limit and direct limit contain most of the operations, the proof of presence of the limits is actual problem.

The inverse (direct) limit is not only an important concept in category theory, but also plays an important role in topology, algebra, homology theory etc. To the date, inverse and direct saystems and their limits were defined in different categories. Furthermore, some of their properties were investigated.

Sheng-Gang Li [18] defined inverse (direct) syatem of fuzzy topological spaces and their limits and obtained their properties for the case of category $L-Top$. M. Ghadri and Davvaz [11] introduced direct system and direct limit of H_v-modules. V. Leoreanu [17] proved that the direct limit (inverse limit) of an SHR direct (respectively, inverse) family of SHR semigroups is an SHR semigroup.

In this study, we firstly give the concepts of inverse and direct systems in the category of soft modules and prove that their limits always exist in this category.

2. Preliminaries

In this section, we recall some basic concepts of soft set theory. Throughout this subsection U refers to on initial universe E is a set of parameters, $P(U)$ is the power set of U, and $A \subseteq E$.

Definition 2.1. [8] A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F, A), or as the set of ε-approximate elements of the soft set.

Definition 2.2. [22] For two soft sets (F, A) and (G, B) over U, (F, A) is called soft subset of (G, B) if

(1) $A \subset B$ and
(2) $\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations. This relationship is denoted by $(F, A) \tilde{\subset} (G, B)$.

Similarly, (F, A) is called a soft superset of (G, B) if (G, B) is a soft subset of (F, A). This relationship is denoted by $(F, A) \tilde{\supset} (G, B)$.

Two soft sets (F, A) and (G, B) over $U$are called soft equal if (F, A) is a soft subset of (G, B), and (G, B) is a soft subset of (F, A).

Definition 2.3. [22] The intersection of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \cap B$ and $\forall \varepsilon \in C$, $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$. This is denoted by $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 2.4. [22] If (F, A) and (G, B) are two soft sets, then (F, A) AND (G, B) is denoted $(F, A) \wedge (G, B)$. $(F, A) \wedge (G, B)$ is defined as $(H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$,$\forall(\alpha, \beta) \in A \times B$.

Definition 2.5. [22]The union of two soft sets (F, A) and (G, B) over is the soft set (H, C), where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } \varepsilon \in A - B, \\ G(\varepsilon), & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cup G(\varepsilon) & \text{if } \varepsilon \in A \cup B. \end{cases}$$

This relationship is denoted by $(F, A) \tilde{\cup} (G, B) = (H, C)$.

Throughout this subsection, let M be a left R-module, A be any nonempty set. $F : A \to P(M)$ refer to a set-valued function and the pair (F, A) is a soft set over M.

Definition 2.6. [23] Let (F, A) be a soft set over M.(F, A) is said to be a soft module over M if and only if $F(x) < M$ for all $x \in A$.

Proposition 2.7. [23] *Let (F, A) and (G, B) be two soft modules over M.*

(1) $(F, A) \tilde{\cap} (G, B)$ *is a soft module over M.*
(2) $(F, A) \tilde{\cup} (G, B)$ *is a soft module over M if $A \cap B = \varnothing$.*

Definition 2.8. [23] Let (F, A) and (G, B) be two soft modules over M. Then $(F, A) + (G, B)$ is defined as $(H, A \times B)$, where $H(x, y) = F(x) + G(y)$ for all $(x, y) \in A \times B$.

Proposition 2.9. [23] *Let (F, A) and (G, B) be two soft modules over M. Then $(F, A) + (G, B)$ is soft module over M.*

Definition 2.10. [23] Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B) = (H, A \times B)$ is defined as $H(x, y) = F(x) \times G(y)$ for all $(x, y) \in (A \times B)$.

Proposition 2.11. [23] *Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B)$ is soft module over $M \times N$.*

Direct product and direct sum are same when the dimension is finite, so " $\oplus$ " can be used instead of " $\times$ " in the above proposition.

Now, let parameter set of $\{(F_i, A_i)\}_{i \in I}$ be fixed point. We denote fixed point of A_i as a_{0i} and let $F_i(a_{0i}) = 0$. For $A = \prod_{i \in I} A_i$and $M = \underset{i \in I}{\oplus} M_i$, we define the mapping $F : A \to M$ by $F(a) = \underset{i \in I}{\oplus} F(a_i)$, for all $a = \{a_i\} \in A$. Then, (F, A) is a soft module over M[4].

Definition 2.12. [4](F, A) is said to be direct sum of $\{(F_i, A_i)\}_{i \in I}$ and denoted as $\underset{i \in I}{\oplus} (F_i, A_i)$.

The mapping $\varphi_j : A_j \to \prod_{i\in I} A_i$ is defined by $\varphi_j(a_j) = \{a_i\}$ such that if $i \neq j$, then $a_i = a_{0i}$ and if $i = j$, then $a_i = a$. Also for embedding mapping $q_j : M_j \to \bigoplus_{i\in I} M_i$, $(q_j, \varphi_j) : (F_j, A_j) \to (F, A)$ is a soft homomorphism of soft modules [4].

Definition 2.13. [23] Let (F, A) and (G, B) be two soft modules over M. Then (G, B) is soft module of (F, A) if

(1) $B \subset A$ and
(2) $G(x) < F(x),\ \forall x \in B$
this denoted by $(G, B)\widetilde{<}(F, A)$.

Proposition 2.14. [23] *Let (F, A) and (G, B) be two soft modules over M, and (G, B) be soft submodule of (F, A). If $f : M \to N$ is homomorphism of module, then $(f(F), A)$ and $(f(G), B)$ are all soft modules over N, and $(f(G), B)$ is soft submodule of $(f(F), A)$.*

Definition 2.15. [23] Let (F, A) and (G, B) be two soft modules over M and N respectively, $f : M \to N$, $g : A \to B$ be two functions. Then we say that (f, g) is a soft homomorphism if the following conditions are satisfied:

(1) $f : M \to N$ is homomorphism of module;
(2) $g : A \to B$ is a mapping;
(3) $f(F(x)) = G(g(x)),\ \forall x \in A$.

At the same time, we say (F, A) is soft homomorphic to (G, B), which denoted by $(F, A) \simeq (G, B)$.

In this definition, if f is an isomorphism from M to N and g is a one-to-one mapping from A onto B, then we say that (F, A) is a soft isomorphism and that (F, A) is a soft isomorphic to (G, B), this is denoted by $(F, A) \cong (G, B)$.

3. Inverse and direct systems of soft modules

We can say that soft modules and their soft homomorphisms form a category. This category is denoted by $SMod$. In this category, we define operation of product and coproduct.

Let $\{(F_i, A_i)\}_{i\in I}$ be a family of soft modules over the family $\{M_i\}_{i\in I}$ of modules. We define the mapping $F : A \to M$ by $F(\{a_i\}) = \prod_{i\in I} F(a_i)$ for every $\{a_i\} \in A$ where $A = \prod_{i\in I} A_i$, $M = \prod_{i\in I} M_i$. Since $F(a_i) \subset M_i$, $\prod_{i\in I} F(a_i)$ is a submodule of M.Thus, the pair (F, A) is a soft module over M. We denote this soft module by $\prod_{i\in I}(F_i, A_i)$ and $\{(F_i, A_i)\}_{i\in I}$ is said to be the product of family of soft modules.

If $q_{i_0} : \prod_{i\in I} A_i \to A_{i_0}$ and $p_{i_0} : \prod_{i\in I} M_i \to M_{i_0}$ are projection mappings, then $(p_{i_0}, q_{i_0}) : \prod_{i\in I}(F_i, A_i) \to (F_{i_0}, A_{i_0})$ is a soft homomorphism of soft modules. Let $\{(f_i, g_i) : (F_i, A_i) \to (K_i, B_i)\}$ be a family of soft homomorphism of soft modules. Then

$$\left(\prod_{i\in I} f_i, \prod_{i\in I} g_i\right) : \prod_{i\in I}(F_i, A_i) \to \prod_{i\in I}(K_i, B_i)$$

is a soft homomorphism of soft modules. Indeed,

$$(\prod_i f_i) \circ \prod_i F_i = \prod_i (f_i \circ F_i) = \prod_i (K_i \circ g_i) = \prod_i K_i \circ \prod_i g_i.$$

For every $i \in I$, the following diagram of soft modules is commutative.

$$\begin{array}{ccc} \prod\limits_{i\in I}(F_i, A_i) & \overset{(p_i,q_i)}{\longrightarrow} & (F_i, A_i) \\ {\scriptstyle (\Pi f_i, \Pi g_i)}\downarrow & & \downarrow^{(f_i,g_i)} \\ \prod\limits_{i\in I}(K_i, B_i) & \underset{(p'_i,q'_i)}{\longrightarrow} & (K_i, B_i) \end{array}$$

Proposition 3.1. *Product operation is defined in the category of soft modules is a functor.*

Let the family set $\{(F_i, A_i)\}_{i\in I}$ of parameters of soft modules be fixed point sets. The fixed point of the set A_i denoted by a_{0_i} and we suppose that $F_i(a_{0_i}) = 0$. We define the mapping $F : A \to M$ by $F(\{a_i\}) = \bigoplus\limits_{i\in I} F(a_i)$ for every $a = \{a_i\} \in A$ where $A = \prod\limits_{i\in I} A_i$, $M = \bigoplus\limits_{i\in I} M_i$. In this case, (F, A) is a soft module over M. This soft module is said to be a direct sum of the soft modules $\{(F_i, A_i)\}_{i\in I}$ and denoted by $\bigoplus\limits_{i\in I}(F_i, A_i)$.

The mapping $\varphi_j : A_j \to \prod\limits_{i\in I} A_i$ is defined by $\varphi_j(a) = \{a_i\}$, such that $a_i = a_{0_i}$ and if $i \neq j$, then $a_j = a$. If $i_j : M_j \to \bigoplus\limits_i M_i$ is a embedding homomorphism, then

$$(i_j, \varphi_j) : (F_j, A_j) \to (F, A)$$

is a soft homomorphism of soft modules.

Now, let $\{(F_i, A_i)\}_{i\in I}$ and $\{(K_i, B_i)\}_{i\in I}$ be two family of soft modules over the families $\{M_i\}_{i\in I}$ and $\{N_i\}_{i\in I}$ of modules respectively and for every $i \in I$ let $(f_i, g_i) : (F_i, A_i) \to (K_i, B_i)$ be a soft homomorphism of soft modules, where $g_i : (A_i, a_{0_i}) \to (B_i, b_{0_i})$ is a mapping of fixed point sets. That is, $g_i(a_{0_i}) = b_{0_i}$. Then

$$(\bigoplus_{i\in I} f_i, \prod_{i\in I} g_i) : \bigoplus_{i\in I}(F_i, A_i) \to \bigoplus_{i\in I}(K_i, B_i)$$

is a soft homomorphism of soft modules and for every $i \in I$, the following diagram of soft modules is commutative.

$$\begin{array}{ccc} (F_i, A_i) & \longrightarrow & \bigoplus\limits_{i\in I}(F_i, A_i) \\ {\scriptstyle (f_i,g_i)}\downarrow & & \downarrow^{(\oplus f_i, \Pi g_i)} \\ (K_i, B_i) & \underset{(p'_i,q'_i)}{\longrightarrow} & \bigoplus\limits_{i\in I}(K_i, B_i) \end{array}$$

Proposition 3.2. *Direct sum operation is defined in the category of soft modules is a functor.*

Now, let the set of I be directed set and for every $i \in I$, (F_i, A_i) be a soft module over M_i and for every $i < i'$, $(p_i^{i'}, q_i^{i'}) : (F_{i'}, A_{i'}) \to (F_i, A_i)$ be a soft homomorphism of soft modules.

Definition 3.3. If the following conditions are satisfied,

(1) For $i = i'$; $q_i^i : 1_{A_i}$, $p_i^i : 1_{M_i}$

(2) For $i < i^{'} < i^{''}$; $q_i^{i^{''}} = q_i^{i^{'}} \circ q_{i^{'}}^{i^{''}}, p_i^{i^{''}} = p_i^{i^{'}} \circ p_{i^{'}}^{i^{''}}$

then the family

$$\left(\{(F_i, A_i)\}_{i \in I}, \{p_i^{i^{'}}, q_i^{i^{'}}\}_{i<i'}\right) \tag{3.1}$$

is said to be inverse system of soft modules.

Theorem 3.4. *Every inverses system in respresentation 3.1 has a limit in the category of $SMod$ and this limit is unique.*

Proof. By definition, the inverse system 3.1 of soft modules consists of the inverse system 3.2 sets

$$\left(\{A_i\}_{i \in I}, \{q_i^{i^{'}}\}_{i<i'}\right) \tag{3.2}$$

and consists of the inverse system 3.3 of modules

$$\left(\{M_i\}_{i \in I}, \{p_i^{i^{'}}\}_{i<i'}\right). \tag{3.3}$$

Let the inverse limit of the system 3.2 be a $A = \varprojlim_i A_i$ and the inverse limit of the system 3.3 be a $M = \varprojlim_i M_i$. We define the mapping $F : A \to \prod_{i \in I} M_i$. The equality $q_i^{i^{'}}(a_{i'}) = a_i$ is satisfied for every $a = \{a_i\} \in A$ and every $i < i^{'}$. Since $(p_i^{i^{'}}, q_i^{i^{'}}) : (F_{i'}, A_{i'}) \to (F_i, A_i)$ is a soft homomorphism, $p_i^{i^{'}}(F_{i'}(a_{i'})) = F_i(q_i^{i^{'}}{}_{i'}(a_{i'})) = F_i(a_i)$ is satisfied. That is,

$$p_i^{i^{'}} |_{F_{i'}(a_{i'})}: F_{i'}(a_{i'}) \to F_i(a_i)$$

Then the family $\left[\{F_i(a_i)\}_{i \in I}, \{p_i^{i^{'}} |_{F_{i'}(a_{i'})}\}_{i<i'}\right]$ forms a inverse system of submodule for every $a = \{a_i\} \in A$. Limit of this system denoted by $\varprojlim_i F_i(a)$.

Now, we define the mapping $F : A \to \prod_{i \in I} M_i$ by $F(a) = \varprojlim_i F(a_i)$ for every $a = \{a_i\} \in A$. Since the module $F(a) = \varprojlim_i F(a_i)$ is a submodule of the module $M = \varprojlim_i M_i$ for every $a \in A$, the pair (F, A) is soft module over the module M.

We show that the soft module (F, A) is a limit of this system. Let (H, B) be any soft module over the module N and $(p_i^{i^{'}}, q_i^{i^{'}}) \circ (h_{i'}, \varphi_i) = (h_i, \varphi_i)$ is satisfied for the homomorphisms of the soft modules $\{(h_i, \varphi_i) : (H, B) \to (F_i, A_i)\}_{i \in I}$ every $i < i^{'} \in I$.

We define the soft homomorphism $(\psi, \gamma) : (H, B) \to \varprojlim_i (F_i, A_i)$. The mapping $\gamma : B \to A = \varprojlim_i A_i$ defined by $\gamma(b) = \{\varphi_i(b)\}$ and the homomorphism $\psi : N \to \varprojlim_i M_i$ defined by $\psi(x) = \{h_i(x)\}$. Then (ψ, γ) is a soft homomorphism of soft modules and for the soft homomorphisms $(\pi_i, q_i) : \varprojlim_i (F_i, A_i) \to (F_i, A_i), q_i : \varprojlim_i A_i \to A_i$, $\pi_i : \varprojlim_i M_i \to M_i$, the following diagram is commutative.

$$\begin{array}{ccc} (H,B) & \xrightarrow{(h_i,\varphi_i)} & (F_i,A_i) \\ {\scriptstyle(\psi,\gamma)}\downarrow & \nearrow_{(\pi_i,q_i)} & \\ \varprojlim_i (F_i,A_i) & & \end{array}$$

Thus, the theorem is proved. □

Now, let the family $\{(K_j, B_j)\}_{j\in J}$ be a family of soft modules over family of the modules $\{N_j\}_{j\in J}$ and

$$\left[\{(K_j, B_j)\}_{j\in J}, \{r_j^{j'}, t_j^{j'}\}_{j<j'}\right]$$

be a inverse system of these soft modules. Let $\varphi : J \to I$ be isoton and surjective mapping, and

$$(f_i, g_i) : (F_{\varphi(j)}, A_{\varphi(j)}) \to (K_j, B_j) \tag{3.4}$$

be a soft homomorphism of soft modules for every $j \in J$.

Definition 3.5. If the following diagram of soft modules is commutative for every $j < j'$,

$$\begin{array}{ccc} (F_{\varphi(j')}, A_{\varphi(j')}) & \longrightarrow & (K_{j'}, B_{j'}) \\ \downarrow & & \downarrow \\ (F_{\varphi(j)}, A_{\varphi(j)}) & \longrightarrow & (K_j, B_j) \end{array}$$

then the family $(\varphi, \{f_j, g_j\}_{j\in J})$ is said to be morphism from the system 3.1 to the system 3.4.

The inverse system of soft modules and their morphism forms a category. This category denoted by $Inv(SMod)$.

Let the family $(\varphi, \{f_j, g_j\}_{j\in J})$ be a morphism of inverse systems. Then the following diagram is commutative.

$$\begin{array}{ccc} \Pi A_{\varphi(j)}) & \xrightarrow{\Pi F_{\varphi(j)}} & \prod_j M_{\varphi(j)} \\ {\scriptstyle \Pi g_i}\downarrow & & \downarrow{\scriptstyle \Pi f_j} \\ \Pi\ B_j & \xrightarrow[\Pi K_j]{} & \prod_j N_j \end{array}$$

$\left(\varphi, \{f_j\} : \{F_{\varphi(j)}(a'_{\varphi(j)})\} \to \{K_j(g_j(a'_j))\}\right)$ is a morphism of inverse systems of modules for every $\{a_{\varphi(j)}\} \in \Pi A_{\varphi(j)})$. Then $\varprojlim(\varphi, \{f_j\}) : \varprojlim_j F_{\varphi(j)}(a'_{\varphi(j)}) \to \varprojlim_j K_j(g_j(a'_j)$ is a homomorphism of limit modules and the following diagram is commutative.

$$\begin{array}{ccc}
\varprojlim_{j} A_{\varphi(j)}) & \overset{\varprojlim F_{\varphi(j)}}{\longrightarrow} & \varprojlim_{j} M_{\varphi(j)} \\
{\scriptstyle \varprojlim g_i}\downarrow & & \downarrow{\scriptstyle \varprojlim_{j} f_j} \\
\varprojlim_{j}\ B_j & \underset{\varprojlim K_j}{\longrightarrow} & \varprojlim_{j} N_j
\end{array}$$

That is, $(\varprojlim f_j, \varprojlim g_i) : \varprojlim(F_{\varphi(j)}, A_{\varphi(j)}) \to \varprojlim(K_j, B_j)$ is a soft homomorphism of soft modules. This homomorphism is said to be limit of the family $(\varphi, \{f_j, g_j\}_{j\in J})$ and denoted by $\varprojlim\,(\varphi, \{f_j, g_j\}_{j\in J})$.

Theorem 3.6. *The corresponding* $\left(\{(F_i, A_i)\}_{i\in I}, \{p_i^{i'}\}_{i<i'}\right) \longmapsto \varprojlim_{i}(F_i, A_i)$ *is a covariant functor from the category of inverse systems of soft modules to the category of soft modules.*

The soft homomorphism $(p_i, q_i) : \prod_{i\in I}(F_i, A_i) \to (F_i, A_i)$ of soft modules define the soft homomorphism $(\pi_i, \overline{q_i}) : \varprojlim_{i}(F_i, A_i) \to (F_i, A_i)$ and for every $i < i' \in I$, the following diagram is commutative.

$$\begin{array}{ccc}
 & & (F_{i'}, A_{i'}) \\
 & \overset{(\pi_{i'}, \overline{q}_{i'})}{\nearrow} & \\
\varprojlim(F_i, A_i) & & \downarrow{\scriptstyle (p_i^{i'}, q_i^{i'})} \\
 & \underset{(\pi_i, \overline{q}_i)}{\searrow} & \\
 & & (F_i, A_i)
\end{array}$$

Let $(F_1, A_1), (F_2, A_2)$ be soft modules over the module M and $(G_1, B_1), (G_2, B_2)$ be soft modules over N, $(f, g) : (F_1, A_1) \to (G_2, B_2), (h, p) : (G_1, B_1) \to (G_2, B_2)$ be the homomorphisms of soft modules and let $A_1 \cap B_1 = A_2 \cap B_2 = \oslash$ be satisfied. In this case, we can look at the soft modules $(F_1, A_1)\widetilde{\cup}(G_1, B_1) = (H_1, C_1)$ and $(F_2, A_2)\widetilde{\cup}(G_2, B_2) = (H_2, C_2)$. In this place, $C_1 = A_1 \cup B_1$ and $C_2 = A_2 \cup B_2$ is satisfied. Since $A_1 \cap B_1 = A_2 \cap B_2 = \oslash$, the mapping $\varphi : C_1 \to C_2$ is defined as follows.

$$\varphi(c) = \begin{cases} g(c) & c \in A_1 \\ p(c) & c \in B_1 \end{cases}$$

The homomorphism $\psi : M \to M$ is defined by $\psi(x) = f(x)$ for $c \in A_1$ and is defined by $\psi(x) = h(x)$ for $c \in A_2$. Then

$$(\psi, \varphi) : (F_1, A_1)\widetilde{\cup}(G_1, B_1) \to (F_2, A_2)\widetilde{\cup}(G_2, B_2)$$

is a soft homomorphism of soft modules.

Proposition 3.7. *The operation* $\widetilde{\cup} : SMod \times SMod \to SMod$ *is s covariant functor.*

Now, let

$$(F,A)=\Big(\{(F_i,A_i)\}_{i\in I},\{p_i^{i'},q_i^{i'}\}_{i<i'}\Big)$$

and

$$(G,B)=\Big(\{(G_i,B_i)\}_{i\in I},\{r_i^{i'},t_i^{i'}\}_{i<i'}\Big)$$

be inverse system of soft modules over family of the modules $\{M_i\}_{i\in I}$ and $(f_i,g_i):(F,A)\to(G,B)$ be morphism of these inverse systems and let $A_i\cap B_i=\oslash$ be for every $i\in I$.We obtain the following inverse system of soft modules.

$$\Big[\{(F_i,A_i)\widetilde{\cup}(G_i,B_i)\}_{i\in I},\{p_i^{i'},r_i^{i'}\}_{i<i'}\Big] \tag{3.5}$$

Theorem 3.8. $\varprojlim\limits_i((F_i,A_i)\widetilde{\cup}(G_i,B_i))=\left(\varprojlim\limits_i(F_i,A_i)\widetilde{\cup}\varprojlim\limits_i(G_i,B_i)\right).$

Proof. The parameter set of the soft module $\varprojlim\limits_i(F_i,A_i)\widetilde{\cup}(G_i,B_i)$ is $\varprojlim\limits_{i\in I}(A_i\cup B_i)$. Since $A_i\cap B_i=\oslash$, $\varprojlim A_i\cup\varprojlim B_i=\varprojlim\limits_{i\in I}(A_i\cup B_i)$ is satisfied. Since $\prod\limits_{i\in I}A_i\cap\prod\limits_{i\in I}B_i$ is satisfied, $\varprojlim A_i\cap\varprojlim B_i=\oslash$.Then c elements $\varprojlim\limits_{i\in I}A_i$ or $\varprojlim\limits_{i\in I}B_i$ for every $c=\{c_i\}\in(\varprojlim\limits_{i\in I}A_i)\cup(\varprojlim\limits_{i\in I}B_i)$.

Let $c=\{c_i\}\in(\varprojlim\limits_{i\in I}A_i)$ be. In this case, the following inverse systems of modules is obtained from the inverse system 3.5.

$$\Big(\{F_i(c_i)\}\{p_i^{i'}\}_{i<i'}\Big) \tag{3.6}$$

Then for $c=\{c_i\}\in(\varprojlim\limits_{i\in I}A_i)$

$$\left(\varprojlim_i(F_i,A_i)\widetilde{\cup}(G_i,B_i)\right)(\{c_i\})=\varprojlim F_i(c_i). \tag{3.7}$$

The theorem is proved from 3.6 and 3.7. □

Theorem 3.9. *If* $\{(F,A)_j\}_{j\in J}$ *is a family of inverse systems of soft modules, then* $\varprojlim\prod\limits_{j\in J}(F,A)_j=\prod\limits_{j\in J}\varprojlim(F,A)_j$ *is satisfied.*

Proof. The proof is obvious. □

Now, We give direct systems and limits of these systems in the category of $SMod$.

Let the family $\{(F_i,A_i)\}_{i\in I}$ be a family soft modules over the family $\{M_i\}_{i\in I}$ and $(p_i^{i'},q_i^{i'}):(F_i,A_i)\to(F_{i'},A_{i'})$ a soft homomorphism of soft modules for every $i<i'$ and let $\{A_i\}$ be a family of sets with fixed point.

Definition 3.10. If the following conditions are satisfied,

(1) $(p_i^{i'},q_i^{i'})=1_{(F_i,A_i)}$

(2) For $i < i^{'} < i^{''}$; $(p_{i^{'}}^{i^{''}} , q_{i^{'}}^{i^{''}}) \circ (p_i^{i^{'}}, q_i^{i^{'}}) = (p_i^{i^{''}}, q_i^{i^{''}})$
then the family

$$\left(\{(F_i, A_i)\}_{i\in I}, \{p_i^{i^{'}}, q_i^{i^{'}}\}_{i<i'}\right) \tag{3.8}$$

is said to be direct system of soft modules.

$\left(\{M_i\}_{i\in I}, \{p_i^{i^{'}}\}_{i<i'}\right)$ is a direct system of modules for the direct system 3.8.

Let $M = \varinjlim_i M_i$. The set $A \subset \prod_{i\in I} A_i$ is defined by $A = \left\{\{a_i\} \mid p_i^{i^{'}}(a_i) = a_{i'}\right\}$.

Theorem 3.11. *Every direct system in respresentation 3.8 has a limit in the category of SMod and this limit is unique.*

Proof. Since $p_i^{i^{'}}(a_i) = a_{i'}$,$\left(\{F(a_i)\}_{i\in I}, q_i^{i^{'}} : F(a_i) \to F(a_{i'})_{i<i'}\right)$ is a direct system of modules for every $a = \{a_i\} \in A$.

We define the mapping $F : A \to M$ by the formule $F(\{a_i\}) = \varinjlim_i F(a_i)$. Clearly, the module $F(\{a_i\})$is a submodule of the module M. Thus, the pair (F, A) is a soft module over M. This module is a direct limit of the system 3.8. Let $(\varphi_i, \psi_i) : (F_i, A_i) \to (H, B)$ which is satisfied the condition $(\varphi_i, \psi_i) = (\varphi_{i'}, \psi_{i'}) \circ (p_i^{i^{'}}, q_i^{i^{'}})$ for every $i < i^{'}$ be a soft homomorphism of soft modules. $\psi_i(a_i) = \psi_{i'}(a_{i'})$ is satisfied from the condition $\psi_i = \psi_{i'} \circ p_i^{i^{'}}$ for every $a = \{a_i\} \in A$. The mapping $\psi : A \to B$ is defined by $\psi(\{a_i\}) = \psi_i(a_i)$. So, ψ is well-defined. We define the homomorphism $h : \varinjlim F_i(a) \to H(\psi(a))$ by $h([x]) = q_i(x_i)$.Here, $q_i\,[x_i] = [x]$ and $q_i : M_i \to \varinjlim M_i$ are satisfied. Clearly, $(\varphi, \psi) : \varinjlim(F_i, A_i) \to (H, B)$ is a soft homomorphism of soft modules and $(h, \psi) \circ (\pi_i, \varphi_i) = (h_i, \psi_i)$ is satisfied. □

Let $\left(\{(F_i, A_i)\}_{i\in I}, \{(p_i^{i^{'}}, q_i^{i^{'}}) : (F_i, A_i) \to (F_{i'}, A_{i'})\}\right)$ and $\left(\{(G_j, B_j)\}_{j\in J}, \{(r_j^{j^{'}}, t_j^{j^{'}}) : (G_j, B_j) \to (G_{j'}, B_{j'})\}\right)$ be two direct systems of soft modules over $\{M_i\}_{i\in I}$ and $\{N_j\}_{j\in J}$. Then the following diagrams are commutative

$$\begin{array}{ccc} A_i & \xrightarrow{F_i} & P(M_i) \\ {\scriptstyle q_i^{i'}}\downarrow & & \downarrow{\scriptstyle p_i^{i'}} \\ A_{i'} & \xrightarrow[F_{i'}]{} & P(M_{i'}) \end{array} \qquad \begin{array}{ccc} B_j & \xrightarrow{G_j} & P(N_j) \\ {\scriptstyle t_j^{j'}}\downarrow & & \downarrow{\scriptstyle p_i^{i'}} \\ B_{j'} & \xrightarrow[G_{j'}]{} & P(N_{j'}) \end{array}$$

where $p_i^{i^{'}} : M_i \to M_{i'}$,$q_i^{i^{'}} : A_i \to A_{i'} r_j^{j^{'}} : N_j \to N_{j'}, t_j^{j^{'}} : B_j \to B_{j'}$.

Let $\varphi : I \to J$ be izoton and surjective mapping and $(f_i, g_i) : (F_i, A_i) \to G_{\varphi(i)}, B_{\varphi(i)})$ be a soft homomorphism of soft modules. then the following diagram is commutative

$$\begin{array}{ccc} A_i & \xrightarrow{F_i} & P(M_i) \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ B_{\varphi(i)} & \xrightarrow[G_{\varphi(i)}]{} & P(N_{\varphi(i)}) \end{array}$$

If the following diagram is commutative,

$$\begin{array}{ccccc}
A_i & \underrightarrow{F_i} & P(M_i) & & \\
q_i^{i'} \downarrow & \searrow & \downarrow p_i^{i'} & \searrow & \\
 & B_{\varphi(i)} & \longrightarrow & & P(N_{\varphi(i)}) \\
A_{i'} & \longrightarrow & P(M_{i'}) & & \downarrow \\
 & \searrow \ \downarrow^{F_{i'}} & & \searrow & \\
 & B_{\varphi(i')} & \longrightarrow & & P(N_{\varphi(i')})
\end{array}$$

then the family $(\varphi, \{f_i, g_i\}_{i\in I})$ is said to be a morphism of direct systems of soft modules.

The direct system of soft modules and their morphisms form a category. This category is denoted by $Dir(SMod)$.

Theorem 3.12. *The corresponding* $\left(\{(F_i, A_i)\}_{i\in I}, \{p_i^{i'}\}_{i<i'}\right) \longmapsto \varinjlim_i (F_i, A_i)$ *is a covariant functor from the category* $Dir(SMod)$ *to the category* $SMod$.

Proof. $\varinjlim(F_i, A_i)$ and $\varinjlim(G_j, B_j)$ are limits of soft modules where $A \subset \prod_{i\in I} A_i, A = \left\{\{a_i\} \mid q_i^{i'}(a_i) = a_{i'}\right\}$ and $B \subset \prod_{j\in J} B_j, B = \left\{\{b_j\} \mid t_j^{j'}(b_j) = b_{j'}\right\}$.

For $g_i : A_i \to B_{\varphi(i)}, g : \prod_{i\in I} A_i \to \prod_{j\in J} B_j, g(\{a_i\}) = \{g(a_i)\}, f : \varprojlim M_i \to \varprojlim N_j, \varinjlim(F_i, A_i) = (F, A), \varinjlim(G_j, B_j) = (G, B)$, the following diagram is commutative.

$$\begin{array}{ccc}
A & \xrightarrow{F} & \varinjlim M_i \\
{}^{g}\downarrow & & \downarrow^{f} \\
B & \xrightarrow[G]{} & \varinjlim N_j
\end{array}$$

Then $(f, g) : \varinjlim(F_i, A_i) \to \varinjlim(G_{\varphi(i)}, B_{\varphi(i)})$ is a soft homomorphism of soft modules and this soft homomorphism is said to be limit of the family $(\varphi, \{f_i, g_i\}_{i\in I})$ and is denoted by $\varinjlim(\varphi, \{f_i, g_i\}_{i\in I})$.

Since the operation $\widetilde{\cup}$ over soft modules is a functor, $\{(F_i, A_i)\widetilde{\cup}(G_i, B_i)\}_{i\in I}$ is also a direct system of soft modules for the direct systems $\{(F_i, A_i)\}$ and $\{(G_i, B_i)\}$ of soft modules, when the condition $A_i \cap B_i = \oslash$ for every $i \in I$. □

Theorem 3.13. $\varinjlim_i \left((F_i, A_i)\widetilde{\cup}(G_i, B_i)\right) = \left(\varinjlim_i (F_i, A_i)\right) \widetilde{\cup} \left(\varinjlim_i (G_i, B_i)\right)$

Proof. The parameters set of the soft module $\varinjlim_i \left((F_i, A_i)\widetilde{\cup}(G_i, B_i)\right)$ is $A \cup B$ where

$$\begin{aligned}
A &= \left\{\{a_i\} \in \Pi A_i \mid q_i^{i'}(a_{i'}) = a_i\right\} \\
B &= \left\{\{b_i\} \in \Pi B_i \mid r_i^{i'}(b_{i'}) = b_i\right\}
\end{aligned}$$

The parameters set of the soft module $\left(\varinjlim_i (F_i, A_i)\right) \widetilde{\cup} \left(\varinjlim_i (G_i, B_i)\right)$ is also $A \cup B$. For every $i \in I$ since $A_i \cap Bi = \oslash$,$A \cap B = \oslash$.

Now, c elements A or c elements B for every $c \in A \cup B$. If $c = \{c_i\} \in A$ is satisfied then $\left\{(F_i, A_i)\widetilde{\cup}(G_i, B_i)\right\}(c) = \{F_i(c_i)\}$ is a direct system of modules. Here,

$$\varinjlim \left[(F_i, A_i)\widetilde{\cup}(G_i, B_i)\right](c) = \varinjlim F_i(c_i).$$

For $c = \{c_i\} \in A$ the following equality is satisfied

$$\left(\left[\varinjlim (F_i, A_i)\right] \widetilde{\cup} \left[\varinjlim (G_i, B_i)\right]\right)(c) = \varinjlim F_i(c_i).$$

In the same way, if $c \in B$, then the following equalities are satisfied

$$\varinjlim \left[(F_i, A_i)\widetilde{\cup}(G_i, B_i)\right](c) = \varinjlim G_i(c_i)$$

Then

$$\varinjlim_i \left((F_i, A_i)\widetilde{\cup}(G_i, B_i)\right) = \left(\varinjlim_i (F_i, A_i)\right) \widetilde{\cup} \left(\varinjlim_i (G_i, B_i)\right).$$

Thus, the theorem is proved. □

4. Conclusion

In this study by using the operations that are given over soft modules, limits of inverse system and direct system and their properties are investigated.

References

[1] A.Aygunoğlu, H.Aygün, Introduction to fuzzy soft groups, Comput. Math. Appl.58 (2009) 1279-1286.
[2] A.R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, Comput. Math. Appl. No.203 (2007) 412-418.
[3] C.L.Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182-190.
[4] Ç.Gunduz, S. Bayramov, Fuzzy soft modules, International Matematical Forum, Vol. 6,No11 (2011) 517-527.
[5] Ç.Gunduz, S. Bayramov,Inverse and Direct System in Category of Fuzzy Modules,Fuzzy sets, Rough Sets and Multivalued Operations and Aplications, Vol:2 (2011) 11-25.
[6] D. Chen, The parametrization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005) 757-763.
[7] D. Dubois, H, Prade, Fuzyy set and Systems : Theory and Applications, Academic Press, New York, (1980).
[8] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl.37 (1999) 19-31.
[9] F. Feng, Y.B. Jun, Zhao, Soft semirings, Comput. Math. Appl.56 (2008) 2621-2628.
[10] G. J. Klir, T. A. Folger, Fuzzy Sets, Uncertainty and Information, Prentice-Hall, (1988).
[11] Ghadri, M.,Davvaz, B.,Direct system and direct limit of H_v-moduless, Iranian Journal os Science &Technology, Transaction A, 28, A2 (2004) 267-275.
[12] H.Aktaş, N. Çağman, Soft sets and soft group, Information Science 177 (2007) 2726-2735.
[13] H. -J. Zimmermann, Fuzzy Set Theory and Its Applications, Kluwer Academic Publishers, (1993).
[14] L.A. Zadeh, Fuzzy sets, Inform. Control 8, (1965) 338-353.
[15] L.A. Zadeh, Toward a generalized theory of uncertainty (GTU) -an outline, Inform. Sci., 172, (2005) 1-40.

[16] L.Jin-Liang, Y.Rui-Xue, Fuzzy soft sets and fuzzy soft groups, Chinese Control and Decision Conference (2008) 2626-2629.
[17] Leoreanu,V.,Direct limitsand inverse limits of SHR semigroups, Southeast Asian Bulletion of Mathematics, 25, (2001) 421-426.
[18] Li,S.-G., Inverse limits in category $L - Top(I)^1$, Fuzzy sets and systems,108, (1999) 235-241.
[19] Li,S.-G., Inverse limits in category $L - Top(II)^1$, Fuzzy sets and systems,109, (2000) 291-299.
[20] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786-1799.
[21] P. K. Maji and A. R. Roy, An Application of soft sets in a decision making problem, Comput. Math. Appl.44 (2002) 1077-1083.
[22] P:K:Maji, R.Bismas, A.R.Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555-562.
[23] Qiu-Mei Sun , Zi-Liong Zhang, Jing Liu, Soft sets and soft modules, Lecture Notes in Comput. Sci. 5009 (2008) 403-409.
[24] U. Acar, F. Koyuncu, B. Tanay, Soft sets and soft rings, Comput. Math. Appl.59 (2010) 3458-3463.
[25] Z. Pawlak, Rough sets, Int. J. Comput. Sci. 11 (1982) 341-356.

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