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RESEARCH

Radius of γ-spirallikeness of order α of some special functions

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Abstract
In light of the Alexander transformation, the class of spirallike functions is significant. The characteristics of special functions also appear very frequently in Geometric function theory. In this paper, we find the radii of γ-spirallike and convex γ-spirallike of order α of certain special functions.

Keywords γ-Spirallike functions · Radius problem · Special functions · Wright functions · Ramanujan type entire functions

Mathematics Subject Classification 30C45 · 30C15

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions normalized by the condition

$$f(0) = 0 = f'(0) - 1$$

in the disk $\mathbb{D} := \mathbb{D}_1$, where $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. In 1969, Robertson [27] studied the following subclasses (Definition A) of univalent and analytic functions, respectively. It was pointed out that under the well-known Alexander transformation

$$I[f](z) = \int_0^z \frac{f(\zeta)}{\zeta},$$

the univalency of spirallike functions is not preserved.

Definition A Let $f \in \mathcal{A}$. Assume that $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $0 \leq \alpha < 1$.

A. The analytic function f is γ-spirallike of order α if and only if

$$\operatorname{Re}\left(e^{-i\gamma}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\gamma,$$

we denote the class of such functions by $\mathcal{S}_p^{\gamma}(\alpha)$.

B. The class $\mathcal{CS}_p^{\gamma}(\alpha)$ of convex γ-spirallike functions of order α is defined by

$$\operatorname{Re}\left(e^{-i\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)\right) > \alpha\cos\gamma.$$

In 1933, Spacek [28] introduced the class $\mathcal{S}_p^{\gamma}(0)$ using the idea of spirallike curves. Each function in $\mathcal{S}_p^{\gamma}(\alpha)$ is univalent in $\mathbb{D}$, but they do not necessarily be starlike. However, for general values of $\gamma(|\gamma| < \pi/2)$, functions in $\mathcal{CS}_p^{\gamma}(0)$ need not be univalent.

Example 1 The function $f(z) = i(1-z)^i - i \in \mathcal{CS}_p^{\pi/4}(0)$, but not univalent.

Robertson [27] and Pfaltzgraff [24] showed that the function $f \in \mathcal{CS}_p^{\gamma}(0)$ is univalent if $0 < \cos\gamma < 1/2$. For harmonic spirallike functions, readers are urged to read [17, 23]. Note that for $\gamma = 0$, the classes $\mathcal{S}_p^{\gamma}(\alpha)$ and $\mathcal{CS}_p^{\gamma}(\alpha)$ reduce to the classes of starlike and convex functions of order α, given by

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{and} \quad \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha,$$

which we denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively.

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Providing counter examples to verify a certain property associated with a function class may not be easy, see Example 1. However, a radius problem for special functions may serve this purpose. Special functions relationship with several subclasses of univalent functions have been studied by radius problems in [2, 3, 5–10, 18, 19]. Particular interest has been on starlikeness and convexity properties of special functions. The $\mathcal{S}^*(\alpha)$-radius of a normalized analytic function g is to find the constant

$$R_\alpha^\star := \sup\left\{r \in \mathbb{R}^+ : \mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) > \alpha, z \in \mathbb{D}_r\right\}$$

and similarly, $\mathcal{C}(\alpha)$-radius can be defined. The radius constants $R_\alpha^\star$ for Bessel functions were obtained in [1, 3] and for Struve functions in [1, 2]. For similar literature on other special functions, readers are referred to read [1, 2, 6, 10]. In 2022 and 2023, several special functions unified treatment for the Ma-Minda classes [22] appeared in [13, 16].

In view of the Example 1 and motivated by the available literature on special functions radius properties, we propose the following radius problem for several special functions in connection with the class of spirallike functions.

Problem A Let g be the normalized form of a special function. The radius of γ-spirallike of order α of g is defined as

$$R_{sp}(\gamma,\alpha) = \sup\left\{r \in \mathbb{R}^+ : \mathrm{Re}\left(e^{-i\gamma}\frac{zg'(z)}{g(z)}\right) > \alpha\cos\gamma, z \in \mathbb{D}_r\right\}.$$

The radius of convex γ-spirallike of order α is given by

$$R_{sp}^c(\gamma,\alpha) = \sup\left\{r \in \mathbb{R}^+ : \mathrm{Re}\left(e^{-i\gamma}\left(1+\frac{zg''(z)}{g'(z)}\right)\right) > \alpha\cos\gamma, z \in \mathbb{D}_r\right\}.$$

In this paper, we answer Problem A for several well-known special functions. We also show an application of our result in deriving the sufficient conditions to be a member of the class $\mathcal{S}_p^\gamma(\alpha)$ in the sense of Example 1. We denote the radius constant $R_{sp}(\gamma,\alpha,g)$ by $R_{sp}(\gamma,\alpha)$ and similarly for others.

2 Radius of Spirallikeness of Wright functions

Named after E. M. Wright, for $\sigma > -1$ and $z, \rho \in \mathbb{C}$, the generalized Bessel function is given by

$$\Phi(\sigma,\rho,z) = \sum_{n\geq 0}\frac{z^n}{n!\Gamma(n\sigma+\rho)}.$$

The function Φ is entire for $\sigma > -1$. Let $\breve{\zeta}_{\sigma,\rho,n}$ be the n-th positive root of the derivative of the function

$$\Psi_{\sigma,\rho}(z) = z^\rho\Phi(\sigma,\rho,-z^2).$$

Further, let $\zeta_{\sigma,\rho,n}$ be the n-th positive root of $\Phi(\sigma,\rho,-z^2)$. Then the result of Baricz et al. [6, Lemma 1, p. 100] states that for $\sigma,\rho > 0$ we may write

$$\Gamma(\rho)\Phi(\sigma,\rho,-z^2) = \prod_{n\geq 1}\left(1-\frac{z^2}{\zeta_{\sigma,\rho,n}^2}\right), \tag{2.1}$$

and the respective zeros satisfy the following interlacing property:

$$\breve{\zeta}_{\sigma,\rho,n} < \zeta_{\sigma,\rho,n} < \breve{\zeta}_{\sigma,\rho,n+1} < \zeta_{\sigma,\rho,n+1}, \quad (n \geq 1). \tag{2.2}$$

Since $\Phi(\sigma,\rho,-z^2) \notin \mathcal{A}$, we take the following normalized Wright functions:

$$\begin{cases} f_{\sigma,\rho}(z) = \left[z^\rho\Gamma(\rho)\Phi(\sigma,\rho,-z^2)\right]^{1/\rho} \\ g_{\sigma,\rho}(z) = z\Gamma(\rho)\Phi(\sigma,\rho,-z^2) \\ h_{\sigma,\rho}(z) = z\Gamma(\rho)\Phi(\sigma,\rho,-z). \end{cases} \tag{2.3}$$

For brevity, we denote $W_{\sigma,\rho}(z) := \Phi(\sigma,\rho,-z^2)$.

Theorem 1 *Let $\sigma,\rho > 0$. The radius $R_{sp}(\gamma,\alpha)$ for the functions $f_{\sigma,\rho}$, $g_{\sigma,\rho}$ and $h_{\sigma,\rho}$ given by (2.3) are the least positive roots of the equations:*

(i) $rW'_{\sigma,\rho}(r) + \rho(1-\alpha)\cos\gamma\, W_{\sigma,\rho}(r) = 0$
(ii) $rW'_{\sigma,\rho}(r) + (1-\alpha)\cos\gamma\, W_{\sigma,\rho}(r) = 0$
(iii) $\sqrt{r}W'_{\sigma,\rho}(\sqrt{r}) + 2(1-\alpha)\cos\gamma\, W_{\sigma,\rho}(\sqrt{r}) = 0$

situated in $(0,\zeta_{\sigma,\rho,1})$, $(0,\zeta_{\sigma,\rho,1})$, and $(0,\zeta_{\sigma,\rho,1}^2)$, respectively.

Proof By the logarithmic differentiation of (2.3) and using (2.1), we may write

$$\begin{cases} \frac{zf'_{\sigma,\rho}(z)}{f_{\sigma,\rho}(z)} = 1+\frac{1}{\rho}\frac{zW'_{\sigma,\rho}(z)}{W_{\sigma,\rho}(z)} = 1-\frac{1}{\rho}\sum_{n\geq 1}\frac{2z^2}{\zeta_{\sigma,\rho,n}^2-z^2} \\ \frac{zg'_{\sigma,\rho}(z)}{g_{\sigma,\rho}(z)} = 1+\frac{zW'_{\sigma,\rho}(z)}{W_{\sigma,\rho}(z)} = 1-\sum_{n\geq 1}\frac{2z^2}{\zeta_{\sigma,\rho,n}^2-z^2} \\ \frac{zh'_{\sigma,\rho}(z)}{h_{\sigma,\rho}(z)} = 1+\frac{1}{2}\frac{\sqrt{z}W'_{\sigma,\rho}(\sqrt{z})}{W_{\sigma,\rho}(\sqrt{z})} = 1-\sum_{n\geq 1}\frac{z}{\zeta_{\sigma,\rho,n}^2-z}. \end{cases} \tag{2.4}$$

We need to show that the following inequalities for $\alpha \in [0,1)$ and $\gamma \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

$$\begin{aligned} &\mathrm{Re}\left(e^{-i\gamma}\frac{zf'_{\sigma,\rho}(z)}{f_{\sigma,\rho}(z)}\right) > \alpha\cos\gamma, \\ &\mathrm{Re}\left(e^{-i\gamma}\frac{zg'_{\sigma,\rho}(z)}{g_{\sigma,\rho}(z)}\right) > \alpha\cos\gamma \end{aligned} \tag{2.5}$$

and

$$\operatorname{Re}\left(e^{-i\gamma}\frac{zh'_{\sigma,\rho}(z)}{h_{\sigma,\rho}(z)}\right) > \alpha\cos\gamma$$

are valid for $z \in \mathbb{D}_{R_{sp}(\gamma,\alpha;f_{\sigma,\rho})}$, $z \in \mathbb{D}_{R_{sp}(\gamma,\alpha;g_{\sigma,\rho})}$, and $z \in \mathbb{D}_{R_{sp}(\gamma,\alpha;h_{\sigma,\rho})}$, respectively, and each of the above inequalities fails in larger disks. It is known [11] that if $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are such that $|z| \le r < \lambda$, then

$$\operatorname{Re}\left(\frac{z}{\lambda - z}\right) \le \left|\frac{z}{\lambda - z}\right| \le \frac{|z|}{\lambda - |z|}. \tag{2.6}$$

Then the inequality

$$\operatorname{Re}\left(\frac{z^2}{\zeta^2_{\sigma,\rho,n} - z^2}\right) \le \left|\frac{z^2}{\zeta^2_{\sigma,\rho,n} - z^2}\right| \le \frac{|z|^2}{\zeta^2_{\sigma,\rho,n} - |z|^2}$$

holds for every $|z| < \zeta_{\sigma,\rho,1}$. Therefore, from (2.4) and (2.6), we have

$$\begin{aligned}
\operatorname{Re}\left(e^{-i\gamma}\frac{zf'_{\sigma,\rho}(z)}{f_{\sigma,\rho}(z)}\right) &= \operatorname{Re}\left(e^{-i\gamma}\right) - \frac{1}{\rho}\operatorname{Re}\left(e^{-i\gamma}\sum_{n\ge1}\frac{2z^2}{\zeta^2_{\sigma,\rho,n} - z^2}\right)\\
&\ge \cos\gamma - \frac{1}{\rho}\left|e^{-i\gamma}\sum_{n\ge1}\frac{2z^2}{\zeta^2_{\sigma,\rho,n} - z^2}\right|\\
&\ge \cos\gamma - \frac{1}{\rho}\sum_{n\ge1}\frac{2|z|^2}{\zeta^2_{\sigma,\rho,n} - |z|^2}\\
&= \frac{|z|f'_{\sigma,\rho}(|z|)}{f_{\sigma,\rho}(|z|)} + \cos\gamma - 1.
\end{aligned} \tag{2.7}$$

Equality in the each of the above inequalities (2.7) holds when $|z| = r$. Thus, for $r \in \left(0, \zeta_{\sigma,\rho,1}\right)$ it follows that

$$\begin{aligned}
&\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(e^{-i\gamma}\frac{zf'_{\sigma,\rho}(z)}{f_{\sigma,\rho}(z)} - \alpha\cos\gamma\right)\right\}\\
&= \frac{|z|\, f'_{\sigma,\rho}(|z|)}{f_{\sigma,\rho}(|z|)} + (1-\alpha)\cos\gamma - 1.
\end{aligned}$$

Now, the mapping $\Theta : \left(0, \zeta_{\sigma,\rho,1}\right) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}
\Theta(r) &= \frac{rf'_{\sigma,\rho}(r)}{f_{\sigma,\rho}(r)} + (1-\alpha)\cos\gamma - 1\\
&= (1-\alpha)\cos\gamma - \frac{1}{\rho}\sum_{n\ge1}\left(\frac{2r^2}{\zeta^2_{\sigma,\rho,n} - r^2}\right).
\end{aligned}$$

is strictly decreasing since

$$\Theta'(r) = -\frac{1}{\rho}\sum_{n\ge1}\left(\frac{4r\zeta^2_{\sigma,\rho,n}}{\left(\zeta^2_{\sigma,\rho,n} - r^2\right)^2}\right) < 0$$

for all $\rho > 0$. On the other hand, since

$$\lim_{r\searrow 0}\Theta(r) = (1-\alpha)\cos\gamma > 0 \text{ and } \lim_{r\nearrow\zeta_{\sigma,\rho,1}}\Theta(r) = -\infty,$$

in view of the minimum principle for harmonic functions imply that the corresponding inequality for $f_{\sigma,\rho}$ in (2.5) for $\rho > 0$ holds if and only if $z \in \mathbb{D}_{R_{sp}(\gamma,\alpha;f_{\sigma,\rho})}$, where $\mathbb{D}_{R_{sp}(\gamma,\alpha;f_{\sigma,\rho})}$ is the smallest positive root of equation

$$\frac{rf'_{\sigma,\rho}(r)}{f_{\sigma,\rho}(r)} = 1 - (1-\alpha)\cos\gamma$$

which is equivalent to

$$\frac{1}{\rho}\frac{rW'_{\sigma,\rho}(r)}{W_{\sigma,\rho}(r)} = -(1-\alpha)\cos\gamma,$$

situated in $\left(0, \zeta_{\sigma,\rho,1}\right)$. Reasoning along the same lines, proofs of the other parts follows. □

Remark 1 Taking $\gamma = 0$ in Theorem 1 yields [6, Theorem 1].

Corollary 1 *(Sufficient conditions) Let $\sigma, \rho > 0$ and the functions $f_{\sigma,\rho}$, $g_{\sigma,\rho}$, and $h_{\sigma,\rho}$ are given by (2.3).*

(i) Then the function $f_{\sigma,\rho} \in \mathcal{S}_p^{\gamma}(\alpha)$, if

$$\frac{W'_{\sigma,\rho}(1)}{W_{\sigma,\rho}(1)} > \rho\,(\alpha - 1)\cos\gamma.$$

(ii) Then the function $g_{\sigma,\rho} \in \mathcal{S}_p^{\gamma}(\alpha)$, if

$$\frac{W'_{\sigma,\rho}(1)}{W_{\sigma,\rho}(1)} > (\alpha - 1)\cos\gamma.$$

(iii) Then the function $h_{\sigma,\rho} \in \mathcal{S}_p^{\gamma}(\alpha)$, if

$$\frac{W'_{\sigma,\rho}(1)}{W_{\sigma,\rho}(1)} > 2\,(\alpha - 1)\cos\gamma.$$

For our next result, we need to recall the Laguerre–Pólya class.

Definition 2 [2], [12, p. 703], [21] A real entire function L maps real line into itself is said to be in the Laguerre–Pólya class $\mathcal{LP}$, if it can be expressed as follows:

$$L(x) = cx^m e^{-ax^2+\beta x}\prod_{k\ge1}\left(1 + \frac{x}{x_k}\right)e^{-\frac{x}{x_k}},$$

where $c, \beta, x_k \in \mathbb{R}$, $a \ge 0$, $m \in \mathbb{N}\cup\{0\}$, and $\sum {x_k}^{-2} < \infty$.

The class $\mathcal{LP}$ is closed under differentiation and consists of entire functions that may be uniformly approximated by

polynomials with only real zeros on the compact sets of the complex plane.

Now we establish the convex analog of the class of γ-spirallike functions of order α.

Theorem 2 *Let $\sigma, \rho > 0$ and the functions $f_{\sigma,\rho}$, $g_{\sigma,\rho}$, and $h_{\sigma,\rho}$ as given in (2.3). Then*

(i) the radius $R^c_{sp}(\gamma,\alpha)$ of $f_{\sigma,\rho}$ is the smallest positive root of the equation

$$\frac{r\Psi''_{\sigma,\rho}(r)}{\Psi'_{\sigma,\rho}(r)} + \left(\frac{1}{\rho} - 1\right)\frac{r\Psi'_{\sigma,\rho}(r)}{\Psi_{\sigma,\rho}(r)} + (1-\alpha)\cos\gamma = 0,$$

(ii) the radius $R^c_{sp}(\gamma,\alpha)$ of $g_{\sigma,\rho}$ is the smallest positive root of the equation

$$rg''_{\sigma,\rho}(r) + (1-\alpha)\cos\gamma\, g'_{\sigma,\rho}(r) = 0,$$

(iii) the radius $R^c_{sp}(\gamma,\alpha)$ of $h_{\sigma,\rho}$ is the smallest positive root of the equation

$$rh''_{\sigma,\rho}(r) + (1-\alpha)\cos\gamma\, h'_{\sigma,\rho}(r) = 0.$$

Proof We first prove part (i). From (2.1), (2.3) and using the Hadamard representation $\Gamma(\rho)\Psi'_{\sigma,\rho}(z) = \rho z^{\rho-1}\prod_{n\geq 1}\left(1 - \frac{z^2}{\breve{\zeta}^2_{\sigma,\rho,n}}\right)$, (see [6, Eq. 7]), we have

$$\begin{aligned}1 + \frac{zf''_{\sigma,\rho}(z)}{f'_{\sigma,\rho}(z)} &= 1 + \frac{z\Psi''_{\sigma,\rho}(z)}{\Psi'_{\sigma,\rho}(z)} + \left(\frac{1}{\rho} - 1\right)\frac{z\Psi'_{\sigma,\rho}(z)}{\Psi_{\sigma,\rho}(z)}\\ &= 1 - \sum_{n\geq 1}\frac{2z^2}{\breve{\zeta}^2_{\sigma,\rho,n} - z^2} - \left(\frac{1}{\rho} - 1\right)\\ &\quad \sum_{n\geq 1}\frac{2z^2}{\zeta^2_{\sigma,\rho,n} - z^2}\end{aligned}$$

and for $\rho > 1$, using the following inequality of [11]:

$$\left|\frac{z}{y-z} - \lambda\frac{z}{x-z}\right| \leq \frac{|z|}{y-|z|} - \lambda\frac{|z|}{x-|z|}, \quad (x > y > r \geq |z|) \tag{2.8}$$

with $\lambda = 1 - 1/\rho$, we get

$$\left|\frac{zf''_{\sigma,\rho}(z)}{f'_{\sigma,\rho}(z)}\right| \leq -\frac{rf''_{\sigma,\rho}(r)}{f'_{\sigma,\rho}(r)} = -\frac{r\Psi''_{\sigma,\rho}(r)}{\Psi'_{\sigma,\rho}(r)} - \left(\frac{1}{\rho} - 1\right)\frac{r\Psi'_{\sigma,\rho}(r)}{\Psi_{\sigma,\rho}(r)}.$$

Also, using the inequality $||x|-|y|| \leq |x-y|$ and the relation in (2.2), we may see that for $\rho > 0$

$$\left|\frac{zf''_{\sigma,\rho}(z)}{f'_{\sigma,\rho}(z)}\right| \leq -\frac{rf''_{\sigma,\rho}(r)}{f'_{\sigma,\rho}(r)},$$

holds in $|z| = r < \breve{\zeta}_{\sigma,\rho,1}$. Therefore, we have

$$\begin{aligned}&\mathrm{Re}\left(e^{-i\gamma}\left(1 + \frac{zf''_{\sigma,\rho}(z)}{f'_{\sigma,\rho}(z)}\right)\right)\\ &= \mathrm{Re}(e^{-i\gamma})\\ &\quad - \mathrm{Re}\left(e^{-i\gamma}\left(\sum_{n\geq 1}\frac{2z^2}{\breve{\zeta}^2_{\sigma,\rho,n} - z^2} + \left(\frac{1}{\rho} - 1\right)\sum_{n\geq 1}\frac{2z^2}{\zeta^2_{\sigma,\rho,n} - z^2}\right)\right)\\ &\geq \cos\gamma - \left|\sum_{n\geq 1}\frac{2z^2}{\breve{\zeta}^2_{\sigma,\rho,n} - z^2} + \left(\frac{1}{\rho} - 1\right)\sum_{n\geq 1}\frac{2z^2}{\zeta^2_{\sigma,\rho,n} - z^2}\right|\\ &\geq \cos\gamma + \frac{rf''_{\sigma,\rho}(r)}{f'_{\sigma,\rho}(r)}\end{aligned} \tag{2.9}$$

hold for $\rho > 1$. Observe that these inequalities also hold for $\rho > 0$. Equality in the each of the above inequalities (2.9) holds when $|z| = r$. Thus, for $r \in (0, \breve{\zeta}_{\sigma,\rho,1})$ it follows that

$$\begin{aligned}&\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(e^{-i\gamma}\left(1 + \frac{zf''_{\sigma,\rho}(z)}{f'_{\sigma,\rho}(z)}\right) - \alpha\cos\gamma\right)\right\}\\ &\quad = (1-\alpha)\cos\gamma + \frac{|z|\, f''_{\sigma,\rho}(|z|)}{f'_{\sigma,\rho}(|z|)}.\end{aligned}$$

Now, the proof of part (i) follows on similar lines as of Theorem 1.

For the remaining parts, take note that the zeros are real and that the functions $g_{\sigma,\rho}$ and $h_{\sigma,\rho}$ belong to the Laguerre–Polya class $\mathcal{LP}$, which is closed under differentiation. Additionally, the derivatives $g'_{\sigma,\rho}$ and $h'_{\sigma,\rho}$ also do. Assuming that $\tau_{\sigma,\rho,n}$ and $\eta_{\sigma,\rho,n}$, respectively, are the positive zeros of $g'_{\sigma,\rho}$ and $h'_{\sigma,\rho}$, we have the following representations:

$$g'_{\sigma,\rho}(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\tau^2_{\sigma,\rho,n}}\right) \quad \text{and} \quad h'_{\sigma,\rho}(z) = \prod_{n\geq 1}\left(1 - \frac{z}{\eta_{\sigma,\rho,n}}\right),$$

which upon logarithmic differentiation conclude that

$$1 + \frac{zg''_{\sigma,\rho}(z)}{g'_{\sigma,\rho}(z)} = 1 - \sum_{n\geq 1}\frac{2z^2}{\tau^2_{\sigma,\rho,n} - z^2} \quad \text{and}$$

$$1 + \frac{zh''_{\sigma,\rho}(z)}{h'_{\sigma,\rho}(z)} = 1 - \sum_{n\geq 1}\frac{z}{\eta_{\sigma,\rho,n} - z}.$$

Further, reasoning along the same lines as in Theorem 1, the result follows at once. □

Remark 2 Taking $\gamma = 0$ in Theorem 2 yields [6, Theorem 5].

3 Radius of Spirallikeness of Mittag-Leffler functions

Define the transformations A, B, and C which map the set $\{(\frac{1}{x}, y) : x > 1, y > 0\}$ into itself and given by

$$A : (\tfrac{1}{x}, y) \to (\tfrac{1}{2x}, y), \quad B : (\tfrac{1}{x}, y) \to (\tfrac{1}{2x}, x + y),$$
$$C : (\tfrac{1}{x}, y) \to \begin{cases} (\frac{1}{x}, y - 1), & \text{if } y > 1; \\ (\frac{1}{x}, y), & \text{if } 0 < y \leq 1. \end{cases}$$

Let us define the set $W_b = A(W_c) \cup B(W_c)$, where

$$W_c := \left\{ \left(\frac{1}{x}, y \right) : 1 < x < 2, y \in [x - 1, 1] \cup [x, 2] \right\}.$$

Denote by W_i, the smallest set containing W_b and invariant under the transformations A, B, and C. In 1971, Prabhakar [25] introduced the function

$$M(x, y, a, z) := \sum_{n \geq 0} \frac{(a)_n z^n}{n! \Gamma(xn + y)},$$

for $x, y, a > 0$, where $(a)_n = \Gamma(a + n)/\Gamma(a)$ represents the Pochhammer symbol. In particular, Wiman and Mittag-Leffler studied the functions $M(x, y, 1, z)$ and $M(x, 1, 1, z)$, respectively. In 2016, Kumar and Pathan [20] proved that if $(\frac{1}{x}, y) \in W_i$ and $a > 0$, then all zeros of $M(x, y, a, z)$ are real and negative. Now let $\lambda_{x,y,a,n}$ be the n-th positive root of $M(x, y, a, -z^2)$ while $\xi_{x,y,a,n}$ denotes the n-th positive root of the derivative of $z^y M(x, y, a, -z^2)$. Thus, the result of Baricz and Prajapati [5, Lemma 1, p. 121] states that if $(\frac{1}{x}, y) \in W_i$ and $a > 0$, then the function $M(x, y, a, -z^2)$ has infinitely many real roots, and can be written as

$$\Gamma(y) M(x, y, a, -z^2) = \prod_{n \geq 1} \left(1 - \frac{z^2}{\lambda^2_{x,y,a,n}} \right),$$

and the respective zeros are interlaced by the relation as follows:

$$\xi_{x,y,a,n} < \lambda_{x,y,a,n} < \xi_{x,y,a,n+1} < \lambda_{x,y,a,n+1} \quad (n \geq 1).$$

Since $M(x, y, a, -z^2) \notin \mathcal{A}$, we perform the following three normalizations given by

$$\begin{cases} f_{x,y,a}(z) = \left[z^y \Gamma(y) M(x, y, a, -z^2) \right]^{1/y} \\ g_{x,y,a}(z) = z \Gamma(y) M(x, y, a, -z^2) \\ h_{x,y,a}(z) = z \Gamma(y) M(x, y, a, -z). \end{cases} \tag{3.1}$$

For brevity, we write $L_{x,y,a}(z) := M(x, y, a, -z^2)$. Now we mimic the ideas applied in Section 2 to arrive at the following result:

Theorem 3 *Let $(\frac{1}{x}, y) \in W_i$, $a > 0$. Then the radius of γ-spirallikeness of order α for the functions $f_{x,y,a}$, $g_{x,y,a}$, and $h_{x,y,a}$ given by (3.1) are the least positive zeros of the equations:*

$$\begin{aligned} &(i) \;\; r L'_{x,y,a}(r) + y\,(1 - \alpha) \cos\gamma\, L_{x,y,a}(r) = 0 \\ &(ii) \;\; r L'_{x,y,a}(r) + (1 - \alpha) \cos\gamma\, L_{x,y,a}(r) = 0 \\ &(iii) \;\; \sqrt{r} L'_{x,y,a}(\sqrt{r}) + 2\,(1 - \alpha) \cos\gamma\, L_{x,y,a}(\sqrt{r}) = 0 \end{aligned}$$

in $(0, \lambda_{x,y,a,1})$, $(0, \lambda_{x,y,a,1})$, and $(0, \lambda^2_{x,y,a,1})$, respectively.

Proof Using (3.1), we obtain after the logarithmic differentiation:

$$\begin{cases} \frac{z f'_{x,y,a}(z)}{f_{x,y,a}(z)} = 1 - \frac{1}{y} \sum_{n \geq 1} \frac{2z^2}{\lambda^2_{x,y,a,n} - z^2} \\ \frac{z g'_{x,y,a}(z)}{g_{x,y,a}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\lambda^2_{x,y,a,n} - z^2} \\ \frac{z h'_{x,y,a}(z)}{h_{x,y,a}(z)} = 1 - \sum_{n \geq 1} \frac{z}{\lambda^2_{x,y,a,n} - z}. \end{cases} \tag{3.2}$$

We need to show that the following inequalities for $\alpha \in [0, 1)$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\begin{aligned} &\mathrm{Re}\left(e^{-i\gamma} \frac{z f'_{x,y,a}(z)}{f_{x,y,a}(z)} \right) > \alpha \cos\gamma, \\ &\mathrm{Re}\left(e^{-i\gamma} \frac{z g'_{x,y,a}(z)}{g_{x,y,a}(z)} \right) > \alpha \cos\gamma \end{aligned} \tag{3.3}$$

and

$$\mathrm{Re}\left(e^{-i\gamma} \frac{z h'_{x,y,a}(z)}{h_{x,y,a}(z)} \right) > \alpha \cos\gamma$$

are valid for $z \in \mathbb{D}_{R_{sp}(\gamma,\alpha;f_{x,y,a})}$, $z \in \mathbb{D}_{R_{sp}(\gamma,\alpha;g_{x,y,a})}$, and $z \in \mathbb{D}_{R_{sp}(\gamma,\alpha;h_{x,y,a})}$, respectively, and each of the above inequalities fails in larger disks. Since, using (2.6),

$$\mathrm{Re}\left(\frac{z^2}{\lambda^2_{x,y,a,n} - z^2} \right) \leq \left| \frac{z^2}{\lambda^2_{x,y,a,n} - z^2} \right| \leq \frac{|z|^2}{\lambda^2_{x,y,a,n} - |z|^2} \tag{3.4}$$

holds for every $|z| < \lambda_{x,y,a,1}$. Therefore, from (3.2) and (3.4), we have

$$\mathrm{Re}\left(e^{-i\gamma} \frac{z f'_{x,y,a}(z)}{f_{x,y,a}(z)} \right) = \mathrm{Re}\left(e^{-i\gamma} \right) - \frac{1}{y} \mathrm{Re}\left(e^{-i\gamma} \sum_{n \geq 1} \frac{2z^2}{\lambda^2_{x,y,a,n} - z^2} \right)$$

$$\geq \cos\gamma - \frac{1}{y}\left|e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\lambda_{x,y,a,n}^2 - z^2}\right|$$
$$\geq \cos\gamma - \frac{1}{y}\sum_{n\geq 1}\frac{2|z|^2}{\lambda_{x,y,a,n}^2 - |z|^2}$$
$$= \frac{|z|f'_{x,y,a}(|z|)}{f_{x,y,a}(|z|)} + \cos\gamma - 1. \tag{3.5}$$

Equality in the each of the above inequalities (3.5) holds when $|z| = r$. Thus, for $z \in \left(0, \lambda_{x,y,a,1}\right)$ it follows that

$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(e^{-i\gamma}\frac{zf'_{x,y,a}(z)}{f_{x,y,a}(z)} - \alpha\cos\gamma\right)\right\}$$
$$= \frac{|z|f'_{x,y,a}(|z|)}{f_{x,y,a}(|z|)} + (1-\alpha)\cos\gamma - 1.$$

Now, the mapping $\Theta : \left(0, \lambda_{x,y,a,1}\right) \longrightarrow \mathbb{R}$ defined by

$$\Theta(r) = \frac{rf'_{x,y,a}(r)}{f_{x,y,a}(r)} + (1-\alpha)\cos\gamma - 1$$
$$= (1-\alpha)\cos\gamma - \frac{1}{y}\sum_{n\geq 1}\left(\frac{2r^2}{\lambda_{x,y,a,n}^2 - r^2}\right)$$

is strictly decreasing since

$$\Theta'(r) = -\frac{1}{y}\sum_{n\geq 1}\left(\frac{4r\lambda_{x,y,a,n}}{\left(\lambda_{x,y,a,n}^2 - r^2\right)^2}\right) < 0$$

for all $y > 0$. On the other hand, since

$$\lim_{r\searrow 0}\Theta(r) = (1-\alpha)\cos\gamma > 0 \text{ and } \lim_{r\nearrow\lambda_{x,y,a,1}}\Theta(r) = -\infty,$$

in view of the minimum principle for harmonic functions imply that the corresponding inequality for $f_{x,y,a}$ in (3.3) for $y > 0$ holds if and only if $z \in \mathbb{D}_{R_{sp}(\gamma,\alpha;f_{x,y,a})}$, where $R_{sp}(\gamma,\alpha)$ of $f_{x,y,a}$ is the smallest positive root of equation

$$\frac{rf'_{x,y,a}(r)}{f_{x,y,a}(r)} = 1 - (1-\alpha)\cos\gamma$$

which is equivalent to

$$\frac{1}{y}\frac{rL'_{x,y,a}(r)}{L_{x,y,a}(r)} = -(1-\alpha)\cos\gamma,$$

situated in $\left(0, \lambda_{x,y,1}\right)$. This completes the first part. Reasoning along the same lines, proofs of the other parts follows. □

Remark 3 Taking $\gamma = 0$ in Theorem 3 yields [5, Theorem 1].

Corollary 2 *(Sufficient conditions) Let* $(\frac{1}{x}, y) \in W_i$, $a > 0$ *and the functions* $f_{x,y,a}$, $g_{x,y,a}$, *and* $h_{x,y,a}$ *be given by* (3.1).

(i) Then the function $f_{x,y,a} \in \mathcal{S}_p^{\gamma}(\alpha)$, *if*

$$\frac{L'_{x,y,a}(1)}{L_{x,y,a}(1)} > y\,(\alpha-1)\cos\gamma.$$

(ii) Then the function $g_{x,y,a} \in \mathcal{S}_p^{\gamma}(\alpha)$, *if*

$$\frac{L'_{x,y,a}(1)}{L_{x,y,a}(1)} > (\alpha-1)\cos\gamma.$$

(iii) Then the function $h_{x,y,a} \in \mathcal{S}_p^{\gamma}(\alpha)$, *if*

$$\frac{L'_{x,y,a}(1)}{L_{x,y,a}(1)} > 2\,(\alpha-1)\cos\gamma.$$

In the following, we derive the result for the convex analog proceedings on similar lines as Theorem 2.

Theorem 4 *Let* $(\frac{1}{x}, y) \in W_i$ *and* $a > 0$. *Let the functions* $f_{x,y,a}$, $g_{x,y,a}$, *and* $h_{x,y,a}$ *be given by* (3.1).*Then*

(i) the radius $R_{sp}^c(\gamma,\alpha)$ *of* $f_{x,y,a}$ *is the smallest positive root of the equation*

$$rf''_{x,y,a}(r) + (1-\alpha)\cos\gamma\, f'_{x,y,a}(r) = 0,$$

(ii) the radius $R_{sp}^c(\gamma,\alpha)$ *of* $g_{x,y,a}$ *is the smallest positive root of the equation*

$$rg''_{x,y,a}(r) + (1-\alpha)\cos\gamma\, g'_{x,y,a}(r) = 0,$$

(iii) the radius $R_{sp}^c(\gamma,\alpha)$ *of* $h_{x,y,a}$ *is the smallest positive root of the equation*

$$rh''_{x,y,a}(r) - (1-\alpha)\cos\gamma\, h'_{x,y,a}(r) = 0.$$

Remark 4 Taking $\gamma = 0$ in Theorem 4 yields [5, Theorem 3].

4 Radius of Spirallikeness of Ramanujan type entire functions

Ismail and Zhang [14] defined the following entire function of growth order zero for $\beta > 0$, called Ramanujan type entire function

$$A_{\theta}^{(\beta)}(c,z) = \sum_{n\geq 0}\frac{(c;\theta)_n\theta^{\beta n^2}}{(\theta;\theta)_n}z^n,$$

where $\beta > 0, 0 < \theta < 1, c \in \mathbb{C}$, $(c;\theta)_0 = 1$, and $(c;\theta)_k = \prod_{j=0}^{k-1}(1-c\theta^j)$ for $k \geq 1$, which is the generalization of both the Ramanujan entire function $A_\theta(z)$ and Stieltjes–Wigert polynomial $S_n(z;\theta)$ defined as (see [15, 26]):

$$A_\theta(-z) = A_\theta^{(1)}(0,z) = \sum_{n=0}^{\infty} \frac{\theta^{n^2}}{(\theta;\theta)_n} z^n$$

and

$$A_\theta^{(1/2)}(\theta^{-n},z) = \sum_{m=0}^{\infty} \frac{(\theta^{-n};\theta)_m \theta^{m^2/2}}{(\theta;\theta)_m} z^m = (\theta;\theta)_n S_n(z\theta^{(1/2)-n};\theta).$$

Since $A_\theta^{(\beta)}(c,z) \notin \mathcal{A}$, therefore we perform three kinds of normalizations as follows:

$$\begin{cases} f_{\beta,\theta,c}(z) := \left(z^\beta A_\theta^{(\beta)}(-c,-z^2)\right)^{1/\beta} \\ g_{\beta,\theta,c}(z) := z A_\theta^{(\beta)}(-c,-z^2) \\ h_{\beta,\theta,c}(z) := z A_\theta^{(\beta)}(-c,-z), \end{cases} \tag{4.1}$$

where $\beta > 0$, $c \geq 0$, and $0 < \theta < 1$.

Now exercising similarly to the above sections, we obtain the following result:

Theorem 5 *Let $\beta > 0$, $c \geq 0$, and $0 < \theta < 1$. Then the radius $R_{sp}(\gamma,\alpha)$ of the functions $f_{\beta,\theta,c}(z)$, $g_{\beta,\theta,c}(z)$, and $h_{\beta,\theta,c}(z)$ given by (4.1) are the least positive zeros of the equations:*

(i) $r f'_{\beta,\theta,c}(r) + ((1-\alpha)\cos\gamma - 1) f_{\beta,\theta,c}(r) = 0$
(ii) $r g'_{\beta,\theta,c}(r) + ((1-\alpha)\cos\gamma - 1) g_{\beta,\theta,c}(r) = 0$
(iii) $r h'_{\beta,\theta,c}(r) + ((1-\alpha)\cos\gamma - 1) h_{\beta,\theta,c}(r) = 0$

in $(0, \psi_{\beta,\theta,1}(c))$, $(0, \psi_{\beta,\theta,1}(c))$, and $(0, \psi^2_{\beta,\theta,1}(c))$, respectively.

Proof From [10, Lemma 2.1, p. 4-5], it follows that for $\beta > 0$, $c \geq 0$, and $0 < \theta < 1$, the function

$$z \to \Psi_{\beta,\theta,c}(z) := A_\theta^{(\beta)}(-c,-z^2)$$

has infinitely many roots, which are all positive. Let $\psi_{\beta,\theta,n}(c)$ be the n-th positive root of $\Psi_{\beta,\theta,c}(z)$. Then we can also write $\Psi_{\beta,\theta,c}(z)$ as

$$\Psi_{\beta,\theta,c}(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\psi^2_{\beta,\theta,n}(c)}\right). \tag{4.2}$$

Let us now consider the function

$$\Phi_{\beta,\theta,c}(z) := z^\beta \Psi_{\beta,\theta,c}(z) \tag{4.3}$$

and $\xi_{\beta,\theta,n}(c)$ be the n-th positive zero of its derivative. Then $\xi_{\beta,\theta,n}(c)$ interlaces with $\psi_{\beta,\theta,n}(c)$ such that

$$\xi_{\beta,\theta,n}(c) < \psi_{\beta,\theta,n}(c) < \xi_{\beta,\theta,n+1}(c) < \psi_{\beta,\theta,n+1}(c)$$

for $n \geq 1$. Now using (4.1) and (4.2), we have for $\beta > 0$, $c \geq 0$, and $0 < \theta < 1$ that

$$\frac{z f'_{\beta,\theta,c}(z)}{f_{\beta,\theta,c}(z)} = 1 + \frac{1}{\beta}\frac{z\Psi'_{\beta,\theta,c}(z)}{\Psi_{\beta,\theta,c}(z)} = 1 - \frac{1}{\beta}\sum_{n\geq 1}\frac{2z^2}{\psi^2_{\beta,\theta,n}(c) - z^2}; \ (c > 0)$$

$$\frac{z g'_{\beta,\theta,c}(z)}{g_{\beta,\theta,c}(z)} = 1 + \frac{z\Psi'_{\beta,\theta,c}(z)}{\Psi_{\beta,\theta,c}(z)} = 1 - \sum_{n\geq 1}\frac{2z^2}{\psi^2_{\beta,\theta,n}(c) - z^2};$$

$$\frac{z h'_{\beta,\theta,c}(z)}{h_{\beta,\theta,c}(z)} = 1 + \frac{1}{2}\frac{\sqrt{z}\Psi'_{\beta,\theta,c}(\sqrt{z})}{\Psi_{\beta,\theta,c}(\sqrt{z})} = 1 - \sum_{n\geq 1}\frac{z}{\psi^2_{\beta,\theta,n}(c) - z}.$$

Further, with a similar exercise as Theorem 1 the proof follows. □

Following the starlike representation from the proof of Theorem 5, we deduce that

Corollary 3 *(Sufficient conditions) Let $\beta > 0$, $c \geq 0$, $0 < \theta < 1$ and the functions $f_{\beta,\theta,c}(z)$, $g_{\beta,\theta,c}(z)$, and $h_{\beta,\theta,c}(z)$ be given by (4.1).*

(i) Then the function $f_{\beta,\theta,c} \in \mathcal{S}_p^\gamma(\alpha)$, if

$$\frac{\Psi'_{\beta,\theta,c}(1)}{\Psi_{\beta,\theta,c}(1)} > \beta(\alpha - 1)\cos\gamma.$$

(ii) Then the function $g_{\beta,\theta,c} \in \mathcal{S}_p^\gamma(\alpha)$, if

$$\frac{\Psi'_{\beta,\theta,c}(1)}{\Psi_{\beta,\theta,c}(1)} > (\alpha - 1)\cos\gamma.$$

(iii) Then the function $h_{\beta,\theta,c} \in \mathcal{S}_p^\gamma(\alpha)$, if

$$\frac{\Psi'_{\beta,\theta,c}(1)}{\Psi_{\beta,\theta,c}(1)} > 2(\alpha - 1)\cos\gamma.$$

We now prove the convex analog of Theorem 5

Theorem 6 *Let $\beta > 0$, $c \geq 0$, and $0 < \theta < 1$. Let the functions $f_{\beta,\theta,c}(z)$, $g_{\beta,\theta,c}(z)$, and $h_{\beta,\theta,c}(z)$ be given by (4.1). Then*

(i) the radius $R^c_{sp}(\gamma,\alpha)$ of $f_{\beta,\theta,c}(z)$ is the least positive zero of the equation

$$r f''_{\beta,\theta,c}(r) + (1-\alpha)\cos\gamma\, f'_{\beta,\theta,c}(r) = 0,$$

(ii) the radius $R^c_{sp}(\gamma,\alpha)$ of $g_{\beta,\theta,c}(z)$ is the least positive zero of the equation

$$rg''_{\beta,\theta,c}(r)+(1-\alpha)\cos\gamma\, g'_{\beta,\theta,c}(r)=0,$$

(iii) the radius $R^c_{sp}(\gamma,\alpha)$ of $h_{\beta,\theta,c}(z)$ is the least positive zero of the equation

$$rh''_{\beta,\theta,c}(r)+(1-\alpha)\cos\gamma\, h'_{\beta,\theta,c}(r)=0.$$

Proof With logarithmic differentiation, using (4.3) and the infinite product representation of Φ' [10, p. 14-15, also see Eq. 4.6], we can write the convexity expression as follows:

$$\begin{aligned}1+\frac{zf''_{\beta,\theta,c}(z)}{f'_{\beta,\theta,c}(z)}&=1+\frac{z\Phi''_{\beta,\theta,c}(z)}{\Phi'_{\beta,\theta,c}(z)}+\left(\frac{1}{\beta}-1\right)\frac{z\Phi'_{\beta,\theta,c}(z)}{\Phi_{\beta,\theta,c}(z)}\\&=1-\sum_{n\geq1}\frac{2z^2}{\xi^2_{\beta,\theta,n}(c)-z^2}-\left(\frac{1}{\beta}-1\right)\\&\quad\sum_{n\geq1}\frac{2z^2}{\psi^2_{\beta,\theta,n}(c)-z^2}.\end{aligned}$$

Since $(z\Psi_{\beta,\theta,c}(z))'$ and $h'_{\beta,\theta,c}(z)$ belong to $\mathcal{LP}$. We may assume $\gamma_{\beta,\theta,n}(c)$ to be the positive roots of $g'_{\beta,\theta,c}(z)$ and $\rho_{\beta,\theta,n}(c)$ to be the positive roots of $h'_{\beta,\theta,c}(z)$. In fact, the growth order of $g'_{\beta,\theta,c}(z)$ and $\Psi_{\beta,\theta,c}(z)$ is same. Therefore, we may write using their infinite product representation as

$$\begin{aligned}1+\frac{zg''_{\beta,\theta,c}(z)}{g'_{\beta,\theta,c}(z)}&=1-\sum_{n\geq1}\frac{2z^2}{\gamma^2_{\beta,\theta,n}(c)-z^2}\\1+\frac{zh''_{\beta,\theta,c}(z)}{h'_{\beta,\theta,c}(z)}&=1-\sum_{n\geq1}\frac{z}{\rho^2_{\beta,\theta,n}(c)-z}.\end{aligned}$$

A similar exercise as of Theorem 3, completes the proof. □

5 Radius of Spirallikeness of Legendre polynomials

The Legendre differential equation is given by

$$((1-z^2)\xi'(z))'+n(n+1)\xi(z)=0.$$

The Legendre polynomials P_n, $n\in\mathbb{Z}^+$ are the solutions of above equation. With an application of well-known Rodrigues formula, we may write P_n by

$$P_n(z)=\frac{1}{2^n n!}\frac{d^n(z^2-1)^n}{dz^n}.$$

Observe that $P_n(-z)=(-1)^nP_n(z)$. Further, it is known that the odd degree Legendre polynomials $P_{2n-1}(z)$ have only real roots such that

$$0=l_0<l_1<\cdots<l_{n-1}\quad\text{or}\quad -l_1>\cdots>-l_{n-1}.\tag{5.1}$$

Thus, we may normalize the legendre polynomial as follows:

$$\mathcal{P}_{2n-1}(z):=\frac{P_{2n-1}(z)}{P'_{2n-1}(0)}=z+\sum_{k=2}^{2n-1}a_kz^k=a_{2n-1}z\prod_{k=1}^{n-1}(z^2-l_k^2).\tag{5.2}$$

Bulut and Engel [8] derived the radius of uniform starlikeness along with radius of starlikeness and convexity of Legendre polynomials of odd degree. We now derive the radius of spirallikeness.

Theorem 7 *Let $\mathcal{P}_{2n-1}$ be given by (5.2). Then*

(i) the radius $R^c_{sp}(\gamma,\alpha)$ of $\mathcal{P}_{2n-1}$ is the smallest positive root of the equation

$$r\mathcal{P}''_{2n-1}(r)+(1-\alpha)\cos\gamma\,\mathcal{P}'_{2n-1}(r)=0,$$

(ii) the radius $R_{sp}(\gamma,\alpha)$ of the normalized Legendre polynomial of odd degree is given by the smallest positive root of the equation

$$r\mathcal{P}'_{2n-1}(r)+((1-\alpha)\cos\gamma-1)\mathcal{P}_{2n-1}(r)=0.$$

Proof We prove first part and second part follows on same lines. From (5.2), upon the logarithmic differentiation, we have

$$1+\frac{z\mathcal{P}''_{2n-1}(z)}{\mathcal{P}'_{2n-1}(z)}=\frac{z\mathcal{P}'_{2n-1}(z)}{\mathcal{P}_{2n-1}(z)}-\frac{\sum_{k=1}^{n-1}\frac{4l_k^2z^2}{(l_k^2-z^2)^2}}{\frac{z\mathcal{P}'_{2n-1}(z)}{\mathcal{P}_{2n-1}(z)}},$$

where

$$\frac{z\mathcal{P}'_{2n-1}(z)}{\mathcal{P}_{2n-1}(z)}=1-\sum_{k=1}^{n-1}\frac{2z^2}{l_k^2-z^2}.$$

Further, after applying the inequality $||x|-|y||\leq|x-y|$ and using (5.1) for $|z|=r<l_1$, we see that

$$\begin{aligned}&\mathrm{Re}\left(e^{-i\gamma}\left(1+\frac{z\mathcal{P}''_{2n-1}(z)}{\mathcal{P}'_{2n-1}(z)}\right)\right)\\&=\mathrm{Re}\left(e^{-i\gamma}\right)\\&\quad-\mathrm{Re}\left(e^{-i\gamma}\sum_{k=1}^{n-1}\frac{2z^2}{l_k^2-z^2}\right)-\mathrm{Re}\left(e^{-i\gamma}\frac{\sum_{k=1}^{n-1}\frac{4l_k^2z^2}{(l_k^2-z^2)^2}}{1-\sum_{k=1}^{n-1}\frac{2z^2}{l_k^2-z^2}}\right)\end{aligned}$$

$$\geq \cos\gamma - \left| e^{-i\gamma} \sum_{k=1}^{n-1} \frac{2z^2}{l_k^2 - z^2} \right| - \left| e^{-i\gamma} \frac{\sum_{k=1}^{n-1} \frac{4l_k^2 z^2}{(l_k^2 - z^2)^2}}{1 - \sum_{k=1}^{n-1} \frac{2z^2}{l_k^2 - z^2}} \right|$$

$$= \cos\gamma - \sum_{k=1}^{n-1} \frac{2r^2}{l_k^2 - r^2} - \frac{\sum_{k=1}^{n-1} \frac{4l_k^2 r^2}{(l_k^2 - r^2)^2}}{1 - \sum_{k=1}^{n-1} \frac{2r^2}{l_k^2 - r^2}}$$

$$= \cos\gamma + \frac{r\mathcal{P}''_{2n-1}(r)}{\mathcal{P}'_{2n-1}(r)}.$$

Also, with similar reasoning as Theorem 3, result follows. □

Remark 5 Taking $\gamma = 0$ in Theorem 7 yields [8, Theorem 2.2] and [8, Theorem 2.1].

6 Radius of Spirallikeness of Lommel functions

Consider the second-order inhomogeneous Bessel differential equation

$$z^2 w''(z) + z w'(z) + (z^2 - v^2) w(z) = z^{\kappa+1},$$

where $\kappa \pm v \notin \mathbb{Z}^-$. Then a particular solution of the above differential equations is the Lommel function $\mathcal{L}_{\kappa,v}$ of first kind, which is given by

$$\mathcal{L}_{\kappa,v} = \frac{z^{\kappa+1}}{(\kappa - v + 1)(\kappa + v + 1)} {}_1F_2 \left(1; \frac{\kappa - v + 3}{2}, \frac{\kappa + v + 3}{2}; -\frac{z^2}{4}\right),$$

where $\frac{1}{2}(-\kappa \pm v - 3) \notin \mathbb{N}$ and ${}_1F_2$ be the hypergeometric function.

Authors in [1, 2], and Baricz and Yağmur [7] studied the radius of starlikeness and convexity for the normalized Lommel functions expressed in terms of the function $\mathcal{L}_{\kappa-\frac{1}{2},\frac{1}{2}}$:

$$f_{\kappa-\frac{1}{2},\frac{1}{2}}(z), \quad g_{\kappa-\frac{1}{2},\frac{1}{2}}(z), \quad \text{and} \quad h_{\kappa-\frac{1}{2},\frac{1}{2}}(z), \tag{6.1}$$

where $0 \neq \kappa \in (-1, 1)$ and

$$\begin{cases} f_{\kappa,v}(z) = ((\kappa - v + 1)(\kappa + v + 1)\mathcal{L}_{\kappa,v}(z))^{\frac{1}{\kappa+1}}, \\ g_{\kappa,v}(z) = (\kappa - v + 1)(\kappa + v + 1) z^{-\kappa} \mathcal{L}_{\kappa,v}(z), \\ h_{\kappa,v}(z) = (\kappa - v + 1)(\kappa + v + 1) z^{(1-\kappa)/2} \mathcal{L}_{\kappa,v}(\sqrt{z}). \end{cases} \tag{6.2}$$

We denote $\mathcal{L}_{\kappa-\frac{1}{2},\frac{1}{2}} = \mathcal{L}_\kappa$ and also write the functions (6.1) as f_κ, g_κ, and h_κ, respectively.

Theorem 8 *Let $\kappa \in (-1, 1)$, $\kappa \neq 0$. Let the functions f_κ, g_κ, and h_κ be given by (6.1). Then*

(i) the radius $R^c_{sp}(\gamma, \alpha)$ of f_κ is the least positive zero of the equation

$$r f''_\kappa(r) + (1 - \alpha)\cos\gamma\, f'_\kappa(r) = 0, \quad \text{if} \quad \kappa \neq -1/2,$$

(ii) the radius $R^c_{sp}(\gamma, \alpha)$ of g_κ is the least positive zero of the equation

$$r g''_\kappa(r) + (1 - \alpha)\cos\gamma\, g'_\kappa(r) = 0,$$

(iii) the radius $R^c_{sp}(\gamma, \alpha)$ of h_κ is the least positive zero of the equation

$$r h''_\kappa(r) + (1 - \alpha)\cos\gamma\, h'_\kappa(r) = 0.$$

Proof Let us prove the first part. Logarithmic differentiation of (6.2) gives that

$$1 + \frac{z f''_\kappa(z)}{f'_\kappa(z)} = 1 + \frac{z\mathcal{L}''_\kappa(z)}{\mathcal{L}'_\kappa(z)} + \left(\frac{1}{\kappa + \frac{1}{2}} - 1\right) \frac{z\mathcal{L}'_\kappa(z)}{\mathcal{L}_\kappa(z)}. \tag{6.3}$$

Invoking the result [7, Lemma 1], we get that

$$\mathcal{L}_\kappa(z) = \frac{z^{\kappa+\frac{1}{2}}}{\kappa(\kappa+1)} \Phi_0(z) = \frac{z^{\kappa+\frac{1}{2}}}{\kappa(\kappa+1)} \prod_{n\geq 1} \left(1 - \frac{z^2}{\tau^2_{\kappa,n}}\right),$$

where the hypergeometric function $\Phi_t(z) := {}_1F_2\left(1; \frac{\kappa-t+2}{2}, \frac{\kappa-t+3}{2}; -\frac{z^2}{4}\right)$ satisfies the conditions of [7, Lemma 1]. Now let $\tau_{\kappa,n}$ and $\check{\tau}_{\kappa,n}$ be the n-th positive zeros of $\mathcal{L}_\kappa$ and $\mathcal{L}'_\kappa$, respectively, and interlaces for $0 \neq \kappa \in (-1, 1)$ (see [7, Theorem 1]). Further, the proof of [7, Theorem 3] reveals that the entire function $\frac{\kappa(\kappa+1)}{\kappa+\frac{1}{2}} z^{-\kappa+\frac{1}{2}} \mathcal{L}'_\kappa(z)$ has order $1/2$. Therefore, we may write

$$\mathcal{L}'_\kappa(z) = \frac{\kappa + \frac{1}{2}}{\kappa(\kappa+1)} z^{\kappa-\frac{1}{2}} \prod_{n\geq 1} \left(1 - \frac{z^2}{\check{\tau}^2_{\kappa,n}}\right).$$

Now we may rewrite the expression (6.3) as

$$1 + \frac{z f''_\kappa(z)}{f'_\kappa(z)} = 1 - \left(\frac{1}{\kappa + \frac{1}{2}} - 1\right) \sum_{n\geq 1} \frac{2z^2}{\tau^2_{\kappa,n} - z^2} - \sum_{n\geq 1} \frac{2z^2}{\check{\tau}^2_{\kappa,n} - z^2}.$$

To complete the proof, we consider the following three cases:

(1.) Let $\kappa \in (0, 1/2]$. Then in the disk $|z| = r < \check{\tau}_{\kappa,1} < \tau_{\kappa,1}$, an application of the inequality $||x|-|y|| \leq |x-y|$

lead us to

$$\left|\frac{zf_\kappa''(z)}{f_\kappa'(z)}\right| \le \left(\frac{1}{\kappa+\frac{1}{2}}-1\right)\sum_{n\ge1}\frac{2r^2}{\tau_{\kappa,n}^2-r^2}+\sum_{n\ge1}\frac{2r^2}{\breve{\tau}_{\kappa,n}^2-r^2} = -\frac{rf_\kappa''(r)}{f_\kappa'(r)}. \tag{6.4}$$

(2.) Let $\kappa \in (1/2, 1)$. Then invoking the inequality (2.8) with $\lambda = 1 - 1/(\kappa + 1/2)$, we see that

$$\left|\frac{zf_\kappa''(z)}{f_\kappa'(z)}\right| \le -\frac{rf_\kappa''(r)}{f_\kappa'(r)}, \tag{6.5}$$

which is equivalent to (6.4).

(3.) Let $\kappa \in (-1, 0)$. Then in first case we simply substitute κ by $\kappa - 1$, Φ_0 by Φ_1, where $\Phi_1 \in \mathcal{LP}$ and the n-th positive roots $\xi_{\kappa,n}$ and $\breve{\xi}_{\kappa,n}$ of Φ_1 and Φ_1', respectively, interlace with each other. Finally, replacing κ by $\kappa + 1$, we arrive at the desired inequality.

Further, proceeding similarly as of Theorem 1, completes the proof of first part.

For the remaining parts, let $\gamma_{\kappa,n}$ and $\rho_{\kappa,n}$ be the n-th positive roots of g_κ' and h_κ', respectively, and $\gamma_{\kappa,1}, \rho_{\kappa,1} < \tau_{\kappa,1}$. Then one can see that the entire functions g_κ' and h_κ' which have order 1/2 [7, Theorem 3] can be represented for $0 \ne \kappa \in (-1, 1)$ as follows:

$$g_\kappa'(z) = \prod_{n\ge1}\left(1-\frac{z^2}{\gamma_{\kappa,n}^2}\right) \quad \text{and} \quad h_\kappa'(z) = \prod_{n\ge1}\left(1-\frac{z}{\rho_{\kappa,n}^2}\right). \tag{6.6}$$

Now from (6.2) and (6.6), by logarithmic differentiation, we arrive at

$$\begin{cases} 1+\frac{zg_\kappa''(z)}{g_\kappa'(z)} = \frac{1}{2}-\kappa+z\frac{(\frac{3}{2}-\kappa)\mathcal{L}_\kappa'(z)+z\mathcal{L}_\kappa''(z)}{(\frac{1}{2}-\kappa)\mathcal{L}_\kappa(z)+z\mathcal{L}_\kappa'(z)} \\ \qquad\qquad = 1-\sum_{n\ge1}\frac{2z^2}{\gamma_{\kappa,n}^2-z^2} \\ 1+\frac{zh_\kappa''(z)}{h_\kappa'(z)} = \frac{1}{2}\left(\frac{3}{2}-\kappa+\sqrt{z}\frac{(\frac{5}{2}-\kappa)\mathcal{L}_\kappa'(\sqrt{z})+\sqrt{z}\mathcal{L}_\kappa''(\sqrt{z})}{(\frac{3}{2}-\kappa)\mathcal{L}_\kappa(\sqrt{z})+\sqrt{z}\mathcal{L}_\kappa'(\sqrt{z})}\right) \\ \qquad\qquad = 1-\sum_{n\ge1}\frac{z}{\rho_{\kappa,n}^2-z}. \end{cases} \tag{6.7}$$

Now an application of the inequality $||x| - |y|| \le |x - y|$ in the expression (6.7) lead us to

$$\begin{cases} \left|\frac{zg_\kappa''(z)}{g_\kappa'(z)}\right| \le \sum_{n\ge1}\frac{2r^2}{\gamma_{\kappa,n}^2-r^2} = -\frac{rg_\kappa''(r)}{g_\kappa'(r)} \\ \left|\frac{zh_\kappa''(z)}{h_\kappa'(z)}\right| \le \sum_{n\ge1}\frac{r}{\rho_{\kappa,n}^2-r} = -\frac{rh_\kappa''(r)}{h_\kappa'(r)}. \end{cases} \tag{6.8}$$

for $|z| = r < \gamma_{\kappa,1}$ and $|z| = r < \rho_{\kappa,1}$. Further, proceeding with the similar method as in Theorem 1, result follows. □

With similar reasoning as Theorem 1, the proof of the following holds.

Theorem 9 *Let $\kappa \in (-1, 1)$, $\kappa \ne 0$. Then the radius of γ-spirallikeness of order α for the functions f_κ, g_κ, and h_κ given by (6.1) are the least positive zeros of the equations given by*

$$\begin{aligned} &(i)\ rf_\kappa'(r)+((1-\alpha)\cos\gamma-1)f_\kappa(r)=0 \\ &(ii)\ rg_\kappa'(r)+((1-\alpha)\cos\gamma-1)g_\kappa(r)=0 \\ &(iii)\ rh_\kappa'(r)+((1-\alpha)\cos\gamma-1)h_\kappa(r)=0 \end{aligned}$$

in $(0, \tau_{\kappa,1})$, $(0, \tau_{\kappa,1})$, and $(0, \tau_{\kappa,1}^2)$, respectively.

Remark 6 Taking $\gamma = 0$, Theorem 9 reduces to [7, Theorem 3].

7 Radius of Spirallikeness of Struve functions

Consider the second-order inhomogeneous Bessel differential equation given by

$$z^2w''(z)+zw'(z)+(z^2-\delta^2)w(z) = \frac{4\left(\frac{z}{2}\right)^{\delta+1}}{\sqrt{\pi}\Gamma\left(\delta+\frac{1}{2}\right)}.$$

Then a particular solution of the above equation is the Struve function $\mathbf{H}_\delta$ of first kind, which is as follows:

$$\mathbf{H}_\delta(z) := \frac{\left(\frac{z}{2}\right)^{\delta+1}}{\sqrt{\frac{\pi}{4}}\Gamma\left(\delta+\frac{1}{2}\right)}{}_1F_2\left(1;\frac{3}{2},\delta+\frac{3}{2};-\frac{z^2}{4}\right),$$

where $-\delta-\frac{3}{2} \notin \mathbb{N}$ and ${}_1F_2$ is a hypergeometric function. Let us consider the normalized functions given by

$$\begin{cases} U_\delta(z) = \left(\sqrt{\pi}2^\delta\left(\delta+\frac{3}{2}\right)\mathbf{H}_\delta(z)\right)^{\frac{1}{\delta+1}}, \\ V_\delta(z) = \sqrt{\pi}2^\delta z^{-\delta}\Gamma\left(\delta+\frac{3}{2}\right)\mathbf{H}_\delta(z), \\ W_\delta(z) = \sqrt{\pi}2^\delta z^{\frac{1-\delta}{2}}\Gamma\left(\delta+\frac{3}{2}\right)\mathbf{H}_\delta(\sqrt{z}). \end{cases} \tag{7.1}$$

Now we may present our result.

Theorem 10 *Let $|\delta| \le 1/2$. Then the radius $R_{sp}(\gamma, \alpha)$ of the functions U_δ, V_δ, and W_δ given by (7.1) are the least positive zeros of the equations:*

$$\begin{aligned} &(i)\ rU_\delta'(r)+((1-\alpha)\cos\gamma-1)U_\delta(r)=0 \\ &(ii)\ rV_\delta'(r)+((1-\alpha)\cos\gamma-1)V_\delta(r)=0 \\ &(iii)\ rW_\delta'(r)+((1-\alpha)\cos\gamma-1)W_\delta(r)=0 \end{aligned}$$

in $(0, z_{\delta,1})$, $(0, z_{\delta,1})$, and $(0, z_{\delta,1}^2)$, respectively.

Proof Let $z_{\delta,n}$ and $\breve{z}_{\delta,n}$ be the n-th positive roots of $\mathbf{H}_\delta$ and $\mathbf{H}'_\delta$, respectively, which in view of [7, Theorem 2] satisfy the interlacing properties. Now invoking the result [4, Lemma 1], we may represent for $|\delta| \leq 1/2$, the function $\mathbf{H}_\delta$ as follows:

$$\mathbf{H}_\delta(z) = \frac{z^{\delta+1}}{\sqrt{\pi}2^\delta\Gamma(\delta+\frac{3}{2})}\prod_{n\geq 1}\left(1-\frac{z^2}{z_{\delta,n}^2}\right). \tag{7.2}$$

Thus, logarithmic differentiation applied in (7.2) yields that

$$\frac{z\mathbf{H}'_\delta(z)}{\mathbf{H}_\delta(z)} = (\delta+1) - \sum_{n\geq 1}\frac{2z^2}{z_{\delta,n}^2 - z^2}.$$

Now parallel computations similar to the Theorem 1 and appropriately using (7.1), we obtain the desired sharp radius constants. □

Remark 7 Taking $\gamma = 0$ in Theorem 10 gives [2, Theorem 2].

Now we present the radius of convex γ-spirallikeness of order α.

Theorem 11 *Let $|\delta| \leq 1/2$. Let the functions U_δ, V_δ, and W_δ be given by (7.1). Then*

(i) the radius $R^c_{sp}(\gamma,\alpha)$ of U_δ is the least positive zero of the equation

$$rU''_\delta(r) + (1-\alpha)\cos\gamma\, U'_\delta(r) = 0,$$

(ii) the radius $R^c_{sp}(\gamma,\alpha)$ of V_δ is the least positive zero of the equation

$$rV''_\delta(r) + (1-\alpha)\cos\gamma\, V'_\delta(r) = 0,$$

(iii) the radius $R^c_{sp}(\gamma,\alpha)$ of W_δ is the least positive zero of the equation

$$rW''_\delta(r) + (1-\alpha)\cos\gamma\, W'_\delta(r) = 0.$$

Proof Let $z_{\delta,n}$ and $\breve{z}_{\delta,n}$ be the n-th positive roots of $\mathbf{H}_\delta$ and $\mathbf{H}'_\delta$, respectively, which in view of [7, Theorem 2] satisfy the interlacing properties. Employing the result [7, Theorem 4], we may represent for $|\delta| \leq 1/2$, the function $\mathbf{H}'_\delta$ as follows:

$$\mathbf{H}'_\delta(z) = \frac{(\delta+1)z^\delta}{\sqrt{\pi}2^\delta\Gamma(\delta+\frac{3}{2})}\prod_{n\geq 1}\left(1-\frac{z^2}{\breve{z}_{\delta,n}^2}\right). \tag{7.3}$$

Thus, logarithmic differentiation applied in (7.3) yields that

$$1 + \frac{z\mathbf{H}''_\delta(z)}{\mathbf{H}'_\delta(z)} = (\delta+1) - \sum_{n\geq 1}\frac{2z^2}{\breve{z}_{\delta,n}^2 - z^2}. \tag{7.4}$$

The entire functions V'_δ and W'_δ have order $1/2$ follow from [7, Theorem 4]. Now assuming $\eta_{\delta,n}$ and $\sigma_{\delta,n}$ to be the n-th positive roots of V'_δ and W'_δ, respectively, we can write their Hadamard factorizations as given by

$$V'_\delta(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\eta_{\delta,n}^2}\right) \quad \text{and} \quad W'_\delta(z) = \prod_{n\geq 1}\left(1-\frac{z}{\sigma_{\delta,n}^2}\right) \tag{7.5}$$

for $|\delta| \leq 1/2$, where V'_δ and W'_δ belong to the Laguerre–Pólya class and $\eta_{\delta,1}, \sigma_{\delta,1} < z_{\delta,1}$. Further an exercise similar to the Theorem 8 and appropriately using (7.1), (7.4), and (7.5), we obtain the desired sharp radius constants. □

Remark 8 Taking $\gamma = 0$ in Theorem 11 gives [7, Theorem 4].

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Author contributions All Authors contributed equally and reviewed the manuscript

Data availability None.

Declarations

Conflict of interest The authors declare that they have no conflict of interest

Appendix

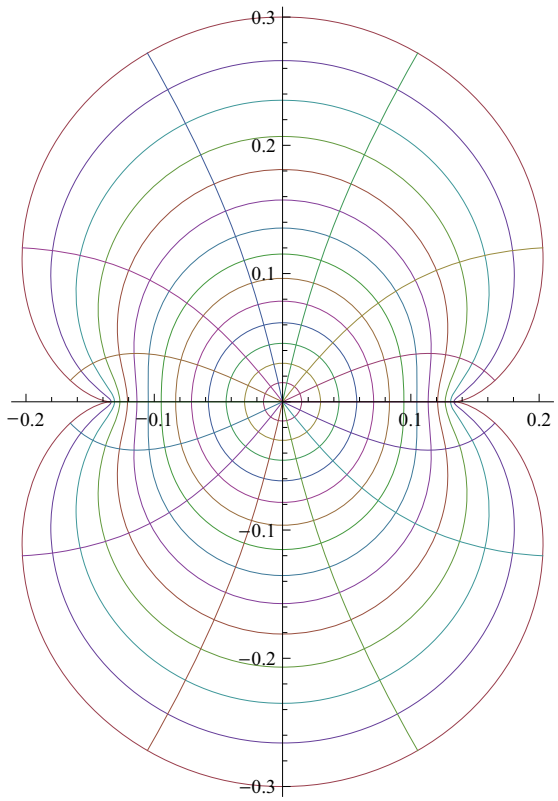
In the following, we demonstrate some of the obtained results numerically and graphically. Tables 1 and 2 demonstrate Theorem 7 and Theorem 11, respectively (Fig. 1).

Table 1 Radii of $\mathcal{P}_{2n-1}$ for $\alpha = 0$ in view of Theorem 7

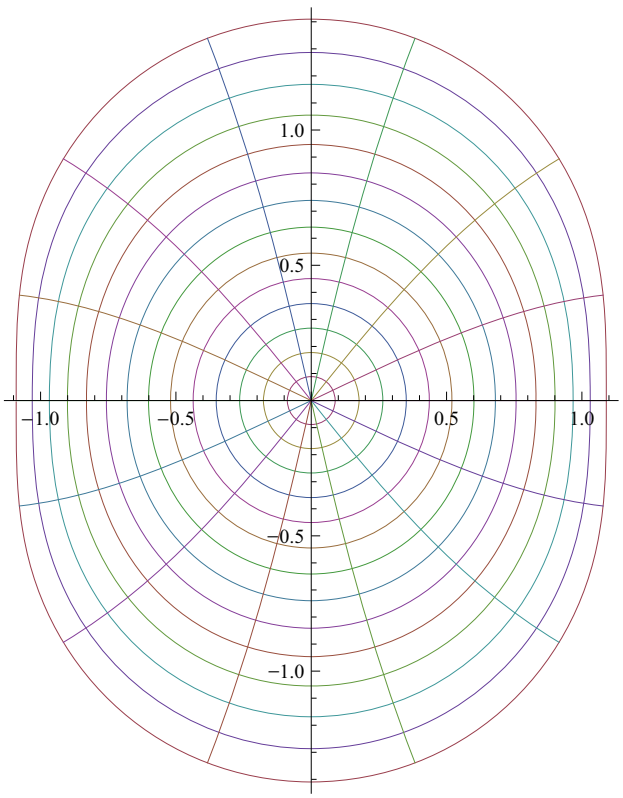
		$n=2$	$n=3$	$n=4$
$\gamma = 0$ and $\alpha = 0$	$R_{sp}(\gamma, \alpha; \mathcal{P}_{2n-1})$	0.4472	0.2852	0.2092
	$R^c_{sp}(\gamma, \alpha; \mathcal{P}_{2n-1})$	0.2581	0.1594	0.1158
$\gamma = \frac{\pi}{6}$ and $\alpha = 0$	$R_{sp}(\gamma, \alpha; \mathcal{P}_{2n-1})$	0.4257	0.2699	0.1977
	$R^c_{sp}(\gamma, \alpha; \mathcal{P}_{2n-1})$	0.2458	0.1513	0.1099
$\gamma = \frac{\pi}{4}$ and $\alpha = 0$	$R_{sp}(\gamma, \alpha; \mathcal{P}_{2n-1})$	0.3958	0.2489	0.1819
	$R^c_{sp}(\gamma, \alpha; \mathcal{P}_{2n-1})$	0.2285	0.1402	0.1017

Table 2 Radii of $U_\delta(z)$, $V_\delta(z)$, and $W_\delta(z)$ for $\delta = \frac{1}{2}$

	$\gamma = 0$ and $\alpha = 0$	$\gamma = \frac{\pi}{6}$ and $\alpha = 0$	$\gamma = \frac{\pi}{4}$ and $\alpha = 0$
$R_{sp}(\gamma, \alpha; U_\delta)$	2.7865	2.6184	2.3936
$R^c_{sp}(\gamma, \alpha; U_\delta)$	1.5053	1.4253	1.3155
$R_{sp}(\gamma, \alpha; V_\delta)$	2.3311	2.1836	1.9886
$R^c_{sp}(\gamma, \alpha; V_\delta)$	1.2417	1.1746	1.0826
$R_{sp}(\gamma, \alpha; W_\delta)$	9.8696	8.7681	7.3814
$R^c_{sp}(\gamma, \alpha; W_\delta)$	4.1158	3.7358	3.2301



(a) Image of function $\mathcal{P}_7$ for $R_{sp} = 0.2092$ when $\gamma = 0$, $\alpha = 0$ and $n = 4$.



(b) Image of function $V_\delta(z)$ for $R^c_{sp} = 1.2417$ when $\delta = \frac{1}{2}$.

Fig. 1 Graphs explaining Theorem 7 and Theorem 11

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