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ORIGINAL RESEARCH PAPER

Geometric properties of functions containing derivatives of Bessel function

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Abstract
Let $J_\xi(z)$ denotes the Bessel function of the first kind of order ξ. For three different kinds of normalizations of $N_\xi(z) = az^2 J''_\xi(z) + bzJ'_\xi(z) + cJ_\xi(z)$, we find the radius for Ma–Minda classes: $\mathcal{S}^*(\varphi)$-radii and $\mathcal{C}(\varphi)$-radii. We further establish the radii of γ-spirallike of order α and convex γ-spirallike of order α of these normalized functions. As an application of our results, we derive sufficient conditions for the $N_\xi(z)$ to be a member of the unified subclasses of starlike and convex functions. The obtained radii are sharp.

Keywords γ-Spirallike functions · Starlikeness and convexity · Bessel functions · Ma–Minda classes · Radius problems

Mathematics Subject Classification 30C45 · 30C15

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions normalized by the condition $f(0) = 0 = f'(0) - 1$ in the unit disk $\mathbb{D} := \mathbb{D}_1$, where $\mathbb{D}_r := \{z \in \mathbb{C}: |z| < r\}$, $0 < r \leq 1$. We say that a function $f \in \mathcal{A}$ is γ-Spirallike of order α if and only if

$$\mathrm{Re}\left(e^{-i\gamma}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\gamma, \quad z \in \mathbb{D},$$

where $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $0 \leq \alpha < 1$. The class of such functions, we denote by $\mathcal{S}_p^\gamma(\alpha)$. In view of the well known Alexender's relation, let $\mathcal{CS}_p^\gamma(\alpha)$ be the class of convex γ-spirallike functions of order α, which is defined below

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$$\mathrm{Re}\left(e^{-i\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)\right) > \alpha\cos\gamma.$$

In 1933, Spacek [29] introduced and studied the class $\mathcal{S}_p^{\gamma}(0)$. Each function in $\mathcal{CS}_p^{\gamma}(\alpha)$ is univalent in $\mathbb{D}$, but they are not necessarily be starlike. Further, it is worth to mention that for general values of $\gamma(|\gamma|<\pi/2)$, a function in $\mathcal{CS}_p^{\gamma}(0)$ need not be univalent in $\mathbb{D}$.

Example 1 [29] The function $f(z)=i(1-z)^i - i \in \mathcal{CS}_p^{\pi/4}(0)$, but not univalent.

Robertson [27] and Pfaltzgraff [26] proved that $f \in \mathcal{CS}_p^{\gamma}(0)$ is univalent if $0<\cos\gamma<1/2$. Note that for $\gamma=0$, the classes $\mathcal{S}_p^{\gamma}(\alpha)$ and $\mathcal{CS}_p^{\gamma}(\alpha)$ reduce to the classes of starlike and convex functions of order α, given by

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{and} \quad \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha,$$

which we denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively.

Question 1 Can we obtain more functions similar to Example 1 using the special function $N_{\xi}(z)$?

We actually answer this with some standard radius problems for special functions. Special functions connections in Geometric function theory have been seen in terms of radius problems in [1–3, 5, 8–10, 15]. In these articles, the $\mathcal{S}^*(\alpha)$-radius of a normalized function f was obtained that is

$$\mathcal{S}^*(\alpha)-radius = \sup\left\{r\in\mathbb{R}^+ : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z\in\mathbb{D}_r\right\}.$$

One may define $\mathcal{C}(\alpha)$-radius similarly. In fact, from the subordination view point, the problem $\mathcal{S}^*(\alpha)$-radius can be generalized to the Ma and Minda [23] subclasses of starlike and convex functions, respectively. The Ma–Minda classes are defined as

$$\mathcal{S}^*(\varphi)=\left\{f\in\mathcal{A}: \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\} \text{ and } \mathcal{C}(\varphi)=\left\{f\in\mathcal{A}: 1+\frac{zf''(z)}{f'(z)} \prec \varphi(z)\right\},$$

where

- φ is analytic and univalent with $\mathrm{Re}\varphi(z)>0,\ \varphi'(0)>0$;
- $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0)=1$ and symmetric about real axis.

Gangania and Kumar [16, 17] gave a unified treatment to the special functions radius of Ma–Minda classes by establishing $\mathcal{S}^*(\varphi)$-radius and $\mathcal{C}(\varphi)$-radius.

Definition 1 Let $f\in\mathcal{A}$ be a special function. Then $\mathcal{S}^*(\varphi)$-radius and $\mathcal{C}(\varphi)$-radius of f are defined as follows:

$$r^*_{\varphi}(f) = \sup\left\{r \in \mathbb{R}^+ : \frac{zf'(z)}{f(z)} \in \varphi(\mathbb{D}),\ z \in \mathbb{D}_r\right\}$$

and

$$r^c_{\varphi}(f) = \sup\left\{r \in \mathbb{R}^+ : 1 + \frac{zf''(z)}{f'(z)} \in \varphi(\mathbb{D}),\ z \in \mathbb{D}_r\right\},$$

respectively.

However, to the best of our knowledge, $\mathcal{S}^{\gamma}_p(\alpha)$-radius and $\mathcal{CS}^{\gamma}_p(\alpha)$-radius for special functions are not handled till date. So, it is natural to inquire the following radius problem in view of Example 1 and known literature on special functions.

Question 2 Let $f \in \mathcal{A}$ be a special function. Then for what radius f belongs to the classes $\mathcal{S}^{\gamma}_p(\alpha)$ and $\mathcal{CS}^{\gamma}_p(\alpha)$, respectively.

We answer Question 2 by considering the following.

Definition 2 Let f in $\mathcal{A}$ be a special function. Then the radius of γ-Spirallike of order α is to find

$$r^*_{sp}(\alpha, \gamma; f) = \sup\left\{r \in \mathbb{R}^+ : \mathrm{Re}\left(e^{-i\gamma}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\gamma, z \in \mathbb{D}_r\right\}$$

and the radius of convex γ-Spirallike of order α is given by

$$r^c_{sp}(\alpha, \gamma; f) = \sup\left\{r \in \mathbb{R}^+ : \mathrm{Re}\left(e^{-i\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \alpha\cos\gamma, z \in \mathbb{D}_r\right\}.$$

Remark 1 One should observe that such radius problem leads to generate a set of examples for functions to be in $\mathcal{S}^{\gamma}_p(\alpha)$ and $\mathcal{CS}^{\gamma}_p(\alpha)$ from the category of special functions in view of Spacek [29].

1.1 Combination of derivative of Bessel function

Recall that the Bessel function of the first kind of order ξ is defined by [25, p. 217]

$$J_{\xi}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\xi+1)}\left(\frac{z}{2}\right)^{2n+\xi} \quad (z \in \mathbb{C}). \tag{1.1}$$

It is known that it has all its zeros real for $\xi > -1$, (see [33, p. 482]). We consider the general function studied by Mercer [24] as

$$N_{\xi}(z) = az^2 J''_{\xi}(z) + bzJ'_{\xi}(z) + cJ_{\xi}(z).$$

As given in [24], we assume $q = b - a$ and ($c = 0$ and $q \neq 0$) or ($c > 0$ and $q > 0$).

From (1.1), we have the power series representation

$$N_\xi(z) = \sum_{n=0}^{\infty} \frac{Q(2n+\xi)(-1)^n}{n!\Gamma(n+\xi+1)} \left(\frac{z}{2}\right)^{2n+\xi} \quad (z \in \mathbb{C}), \tag{1.2}$$

where $Q(\xi) = a\xi(\xi-1) + b\xi + c$ $(a, b, c \in \mathbb{R})$. There are three important works on the function N_ξ. Mercer [24] proved that the kth positive zero of N_ξ increases with ξ in $\xi > 0$. Ismail and Muldoon [18] showed that under the conditions $a, b, c \in \mathbb{R}$ such that $c = 0$ and $b \neq a$ or $c > 0$ and $b > a$:

(i) For $\xi > 0$, the zeros of $N_\xi(z)$ are either real or purely imaginary.
(ii) For $\xi \geq \max\{0, \xi_0\}$, where ξ_0 is the largest real root of the quadratic $Q(\xi) = a\xi(\xi-1) + b\xi + c$, the zeros of $N_\xi(z)$ are real.
(iii) If $\xi > 0$, $\left(a\xi^2 + (b-a)\xi + c\right)/(b-a) > 0$ and $a/(b-a) < 0$, the zeros of $N_\xi(z)$ are all real except for a single pair which are conjugate purely imaginary.

Baricz et al. [6] obtained sufficient and necessary conditions for the starlikeness of a normalized form of N_ξ by using results of Mercer [24], Ismail and Muldoon [18] and Shah and Trimble [28]. For more geometric properties involving the Bessel functions (or generalized Bessel functions), see [31, 32]

Note that N_ξ does not belong to $\mathcal{A}$. For this paper, we consider the following normalizations of the function N_ξ:

$$f_\xi(z) = \left[\frac{2^\xi \Gamma(\xi+1)}{Q(\xi)} N_\xi(z)\right]^{\frac{1}{\xi}}, \tag{1.3}$$

$$g_\xi(z) = \frac{2^\xi \Gamma(\xi+1) z^{1-\xi}}{Q(\xi)} N_\xi(z), \tag{1.4}$$

$$h_\xi(z) = \frac{2^\xi \Gamma(\xi+1) z^{1-\frac{\xi}{2}}}{Q(\xi)} N_\xi(\sqrt{z}). \tag{1.5}$$

In the rest of this paper, for the quadratic $Q(\xi) = a\xi(\xi-1) + b\xi + c$, we shall always assume that $a, b, c \in \mathbb{R}$ $(c = 0$ and $a \neq b)$ or $(c > 0$ and $a < b)$. Moreover, ξ_0 is the largest real root of the quadratic $Q(\xi)$ defined according to the above conditions.

One can see that the functions $N_\xi(z)$ and $N'_\xi(z)$ are entire functions of order zero, then they have infinitely many zeros. According to Hadamard factorization theorem [21]

$$N_\xi(z) = \frac{Q(\xi) z^\xi}{2^\xi \Gamma(\xi+1)} \prod_{n \geq 1} \left(1 - \frac{z^2}{\chi^2_{\xi,n}}\right) \tag{1.6}$$

and

$$N'_{\xi}(z) = \frac{Q(\xi) z^{\xi-1} \xi}{2^{\xi}\Gamma(\xi+1)} \prod_{n\geq 1} \left(1 - \frac{z^2}{\chi'^2_{\xi,n}}\right), \tag{1.7}$$

where $\chi_{\xi,n}$ and $\chi'_{\xi,n}$ denote the nth positive zero of $N_{\xi}(z)$ and $N'_{\xi}(z)$, respectively. In particular, we have

Example 2 One may write the function $N_{\xi}(z)$ in terms of elementary trigonometric functions for the value $\xi = 1/2$ as follows:

$$N_{1/2}(z) = \frac{4(b-a)z\cos z + [a(3-4z^2) - 2b + 4c]\sin z}{2\sqrt{2\pi}\sqrt{z}}.$$

Consequently,

$$f_{1/2}(z) = \frac{[4(a-b)z\cos z + (4az^2 - 3a + 2b - 4c)\sin z]^2}{(a-2b-4c)^2 z},$$
$$g_{1/2}(z) = \frac{4(a-b)z\cos z + (4az^2 - 3a + 2b - 4c)\sin z}{a-2b-4c}$$

and

$$h_{1/2}(z) = \frac{4(a-b)z\cos\sqrt{z} + (4az - 3a + 2b - 4c)\sqrt{z}\sin\sqrt{z}}{a-2b-4c}.$$

For the recent study on the geometric properties of Bessel functions, we refer to read [3–7, 11–14, 20, 22, 30]. However, in context of the Question 2, Kazımoğlu and Deniz [19] obtained the radius of starlikeness and convexity of order α which is nothing but $\mathcal{S}^0_p(\alpha)$-radius for the derivative of Bessel functions.

Theorem A [19, Theorem 2.1] *Let* $\alpha \in [0,1)$. *The following statements holds*:

(a) If $\xi \geq \max\{0, \xi_0\}$, $\xi \neq 0$ then the radius of starlikeness of order α of the function f_{ξ} is the smallest positive positive root of the equation

$$ar^3 J'''_{\xi}(r) + (2a + b - a\xi\alpha)r^2 J''_{\xi}(r) + (b + c - b\xi\alpha)rJ'_{\xi}(r) - c\xi\alpha J_{\xi}(r) = 0.$$

(b) If $\xi \geq \max\{0, \xi_0\}$, then the radius of starlikeness of order α of the function g_{ξ} is the smallest positive positive root of the equation

$$(1-\xi) + \frac{ar^3 J'''_{\xi}(r) + (2a+b)r^2 J''_{\xi}(r) + (b+c)rJ'_{\xi}(r)}{ar^2 J''_{\xi}(r) + brJ'_{\xi}(r) + cJ_{\xi}(r)} = \alpha.$$

(c) If $\xi \geq \max\{0, \xi_0\}$, then the radius of starlikeness of order α of the function h_{ξ} is the smallest positive positive root of the equation

$$(2-\xi)+\frac{ar\sqrt{r}J_{\xi}'''(\sqrt{r})+(2a+b)r^2J_{\xi}''(\sqrt{r})+(b+c)\sqrt{r}J_{\xi}'(\sqrt{r})}{arJ_{\xi}''(\sqrt{r})+b\sqrt{r}J_{\xi}'(\sqrt{r})+cJ_{\xi}(\sqrt{r})}=2\alpha.$$

Let $r_{\alpha}^{c}(f):=r_{sp}^{c}(\alpha,0,f)$ denotes the radius of convexity of order α for f.

Theorem B [19, Theorem 2.2] *Let $\alpha\in[0,1)$. The following statements holds*:

(a) If $\xi\geq\max\{0,\xi_0\}$, $\xi\neq 0$ then the radius $r_{\alpha}^{c}(f_{\xi})$ is the smallest positive positive root of the equation

$$1+\frac{rN_{\xi}''(r)}{N_{\xi}'(r)}+\left(\frac{1}{\xi}-1\right)\frac{rN_{\xi}'(r)}{N_{\xi}(r)}=\alpha.$$

Moreover, $r_{\alpha}^{c}(f_{\xi})<\chi_{\xi,1}'<\chi_{\xi,1}$, where $\chi_{\xi,1}$ and $\chi_{\xi,1}'$ denote the first positive zero of N_{ξ} and N_{ξ}', respectively.

(b) If $\xi\geq\max\{0,\xi_0\}$, then the radius $r_{\alpha}^{c}(g_{\xi})$ is the smallest positive positive root of the equation

$$1+\frac{r^2N_{\xi}''(r)+(2-2\xi)rN_{\xi}'(r)+(\xi^2-\xi)N_{\xi}(r)}{rN_{\xi}'(r)+(1-\xi)N_{\xi}(r)}=\alpha.$$

(c) If $\xi\geq\max\{0,\xi_0\}$, then the radius $r_{\alpha}^{c}(g_{\xi})$ is the smallest positive positive root of the equation

$$1+\frac{r^2N_{\xi}''(\sqrt{r})+(3-2\xi)\sqrt{r}N_{\xi}'(\sqrt{r})+(\xi^2-2\xi)N_{\xi}(\sqrt{r})}{2\sqrt{r}N_{\xi}'(\sqrt{r}+2(2-\xi)N_{\xi}(\sqrt{r}))}=\alpha.$$

We must also observe that Theorems A–B address $\mathcal{S}^*\left(\frac{1+(1-2\alpha)}{1-z}\right)$-radii and $\mathcal{C}\left(\frac{1+(1-2\alpha)}{1-z}\right)$-radii respectively. At this conjunction, we naturally raise the following:

Question 3 Can we generalize Theorem A and Theorem B for the Ma–Minda classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$?

In this paper, we find the radius of γ-Spirallike of order α and the radius of convex γ-Spirallike of order α for the functions $f_{\xi}(z)$, $g_{\xi}(z)$ and $h_{\xi}(z)$ defined by (1.3), (1.4) and (1.5) in the case when $\xi\geq\max\{0,\xi_0\}$. Also, we determine the $\mathcal{S}^*(\varphi)$-radii and $\mathcal{C}(\varphi)$-radii of these functions. Hence, we answer Questions 2 and 3. Consequently, we obtain sufficient conditions on $N_{\xi}(z)$ for its normalized forms to be a member of $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$.

2 Preliminaries

We recall, following [2, 14], that a real entire function ψ belongs to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented in the form

$$\psi(x) = cx^m e^{-ax^2+\beta x} \prod_{k\geq 1}\left(1+\frac{x}{x_k}\right)e^{-\frac{x}{x_k}},$$

with $c, \beta, x_k \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N}\cup\{0\}$ and $\sum x_k^{-2} < \infty$. Similarly, ϕ is said to be of type $\mathcal{I}$ in the Laguerre-Pólya class, written $\varphi \in \mathcal{LPI}$, if $\phi(x)$ or $\phi(-x)$ can be represented as

$$\phi(x) = cx^m e^{\sigma x} \prod_{k\geq 1}\left(1+\frac{x}{x_k}\right),$$

with $c \in \mathbb{R}$, $\sigma \geq 0$, $m \in \mathbb{N}\cup\{0\}$, $x_k > 0$ and $\sum x_k^{-1} < \infty$. The interlacing property of the zeros given below will be used to prove our results.

Lemma 1 [19] *If $\xi \geq \max\{0, \xi_0\}$ then the functions*

$$z \longmapsto \Psi_\xi(z) = \frac{2^\xi \Gamma(\xi+1)}{Q(\xi) z^\xi} N_\xi(z)$$

has infinitely many zeros and all of them are positive. Denoting by $\chi_{\xi,n}$ the nth positive zero of $\Psi_\xi(z)$, under the same conditions the Weierstrassian decomposition

$$\Psi_\xi(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\chi_{\xi,n}^2}\right)$$

is valid, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, if we denote by $\chi'_{\xi,n}$ the nth positive zero of $\Phi'_\xi(z)$, where $\Phi_\xi(z) = z^\xi \Psi_\xi(z)$, then the positive zeros of $\Psi_\xi(z)$ are interlaced with those of $\Phi'_\xi(z)$. In the other words, the zeros satisfy the chain of inequalities

$$\chi'_{\xi,1} < \chi_{\xi,1} < \chi'_{\xi,2} < \chi_{\xi,2} < \chi'_{\xi,3} < \chi_{\xi,3} < \cdots.$$

3 $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$-radius of derivative of Bessel functions

We here generalize Theorems A and B and solve Question 3.

Theorem 3 *Let $\xi \geq \max\{0, \xi_0\}$. The $\mathcal{S}^*(\varphi)$-radii of the functions f_ξ, g_ξ and h_ξ are respectively, given by the smallest positive root of the equations*

(a) $rN'_\xi(r) - (1-\beta)\xi N_\xi(r) = 0$ with $\xi \neq 0$,
(b) $rN'_\xi(r) - (\xi-\beta)N_\xi(r) = 0$,
(c) $\sqrt{r}N'_\xi(\sqrt{r}) - (\xi-2\beta)N_\xi(\sqrt{r}) = 0$

in $|z|<(0,\chi_{\xi,1})$, $(0,\chi_{\xi,1})$ and $(0,\chi^2_{\xi,1})$, where $\varphi(-1)=1-\beta$ and β is the radius of the largest disk $\{w:|w-1|<\beta\}\subseteq\varphi(\mathbb{D})$.

Proof We first prove part (a) : from Eq. (4.2), we have

$$\frac{zf'_\xi(z)}{f_\xi(z)}=\frac{1}{\xi}\frac{zN'_\xi(z)}{N_\xi(z)}=1-\frac{1}{\xi}\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n}-z^2},(\xi\geq\max\{0,\xi_0\},\ \xi\neq 0). \tag{3.1}$$

Let us consider the continuous function $T_{f_\xi}:(0,\chi_{\xi,1})\to\mathbb{R}$

$$T_{f_\xi}(r)=\frac{1}{\xi}\sum_{n\geq 1}\frac{2r^2}{\chi^2_{\xi,n}-r^2}-\beta. \tag{3.2}$$

Then

$$T'_{f_\xi}(r)=\frac{1}{\xi}\sum_{n\geq 1}\left(\frac{4r\chi^2_{\xi,n}}{(\chi^2_{\xi,n}-r^2)^2}\right)>0$$

for all $\xi\geq\max\{0,\xi_0\}\neq 0$ and for $r<\chi_{\xi,1}$. Also, $T_{f_\xi}(0)=-\beta<0$ and $\lim_{r\nearrow\chi_{\xi,1}}T_{f_\xi}(r)=\infty$. This implies that there exist a unique positive root, say $R_\varphi(f_\xi)$ of the equation $T_{f_\xi}(r)=0$ in $(0,\chi_{\xi,1})$. Now let $\{w:|w-1|<\beta\}\subseteq\varphi(\mathbb{D})$ such that $\varphi(-1)=1-\beta$. Therefore, using the Lemma [13] that if $z\in\mathbb{C}$ and $\chi\in\mathbb{R}$ are such that $|z|\leq r<\chi$, then

$$\mathrm{Re}\left(\frac{z}{\chi-z}\right)\leq\left|\frac{z}{\chi-z}\right|\leq\frac{|z|}{\chi-|z|},$$

in (3.1), we get that

$$\left|\frac{zf'_\xi(z)}{f_\xi(z)}-1\right|=\left|\frac{1}{\xi}\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n}-z^2}\right|\leq\frac{1}{\xi}\sum_{n\geq 1}\frac{2r^2}{\chi^2_{\xi,n}-r^2},\leq\beta \tag{3.3}$$

which implies that $f_\xi\in\mathcal{S}^*(\varphi)$ in $|z|<R_\varphi(f_\xi)$. Further, taking $z\in\mathbb{D}$ such that $z=r=-R_\varphi(f_\xi)$, from (3.2) and (3.3), it follows that

$$\left|\frac{zf'_\xi(z)}{f_\xi(z)}-1\right|=\beta,$$

which implies that for $zf'_\xi(z)/f_\xi(z)\notin\varphi(\mathbb{D})$ for all $|z|=r\geq R_\varphi(f_\xi)$, and this proves the sharpness part. Reasoning along the same lines, proofs of the parts $b)$ and $c)$ follow. □

The Fig. 1 was created by taking $a=1$, $b=2$, $c=0$ and $\varphi(z)=e^z$ in the Theorem 3.

Remark 2 Theorem 3 with $\varphi(z)=(1+(1-2\beta)z)/(1-z)$ coincide with Theorem 5 with $\alpha=\beta$ and $\gamma=0$. In particular, choosing $\varphi(z)=(1+z)/(1-z)$ and $\beta=1$ in Theorem 3 reduces to Theorem 5 with $\alpha=0$ and $\gamma=0$.

As a consequence of Theorem 3, we get the sufficient conditions for the functions to belong in the class $\mathcal{S}^*(\varphi)$.

Corollary 1 (Sufficient conditions for Ma–Minda starlikeness) *Let* $\xi \geq \max\{0, \xi_0\}$. *Further,* $\varphi(-1) = 1 - \beta$ *and* β *is the radius of the largest disk* $\{w : |w - 1| < \beta\} \subseteq \varphi(\mathbb{D})$. *Then*

(i) if $1 - \frac{1}{\xi}\frac{N'_\xi(1)}{N_\xi(1)} < \beta$ and $\xi \neq 0$, then the function f_ξ belongs to $\mathcal{S}^*(\varphi)$,

(ii) if $\xi - \frac{N'_\xi(1)}{N_\xi(1)} < \beta$, then the function g_ξ belongs to $\mathcal{S}^*(\varphi)$,

(iii) if $\frac{\xi}{2} - \frac{N'_\xi(1)}{2N_\xi(1)} < \beta$, then the function h_ξ belongs to $\mathcal{S}^*(\varphi)$.

In the following, we prove the result for the Ma–Minda class of convex functions, which generalize Theorem B.

Theorem 4 *The* $\mathcal{C}(\varphi)$*-radii of the functions* f_ξ, g_ξ *and* h_ξ *are respectively, given by the smallest positive root of the equations*

(a) $rf''_\xi(r) + \beta\xi f'_\xi(r) = 0$, where $\xi \geq \max\{0, \xi_0\}$, $\xi \neq 0$,

(b) $rg''_\xi(r) + \beta g'_\xi(r) = 0$, where $\xi \geq \max\{0, \xi_0\}$,

(c) $\sqrt{r}h''_\xi(\sqrt{r}) + \beta h'_\xi(\sqrt{r}) = 0$, where $\xi \geq \max\{0, \xi_0\}$

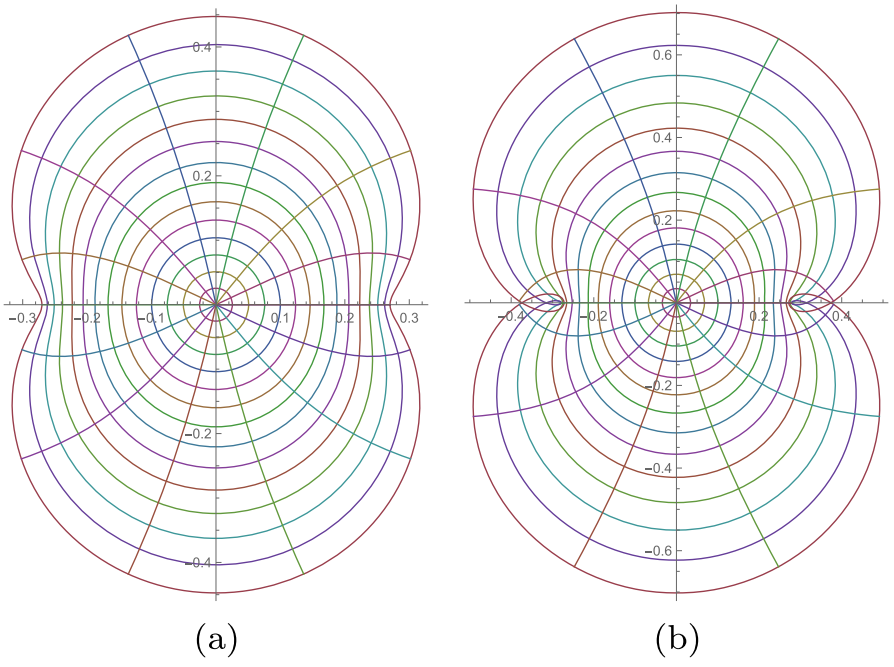


Fig. 1 Images of function $g_{1/2}(z)$ for $r = 0.3571$ and $r = 0.48$, respectively as per Theorem 3 and Table 1

Table 1 Radii of $\mathcal{S}^*(e^z)$ for the functions $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$

	$b=3$ and $c=0$			$a=1$ and $c=0$			$a=1$ and $b=2$		
	$a=2$	$a=3$	$a=4$	$b=2$	$b=3$	$b=4$	$c=2$	$c=3$	$c=4$
$r^*_{e^z}\left(f_{\frac{1}{2}}\right)$	0.2354	0.1818	0.1352	0.2674	0.3050	0.3268	0.4696	0.5272	0.5718
$r^*_{e^z}\left(g_{\frac{1}{2}}\right)$	0.3140	0.2421	0.1798	0.3571	0.4078	0.4372	0.6350	0.7160	0.7792
$r^*_{e^z}\left(h_{\frac{1}{2}}\right)$	0.1611	0.0953	0.0523	0.2090	0.2736	0.3150	0.6845	0.8819	1.0559

in $|z|<(0,\chi_{\xi,1})$, $(0,\chi_{\xi,1})$ and $(0,\chi^2_{\xi,1})$, where $\varphi(-1)=1-\beta$ and β is the radius of the largest disk $\{w:|w-1|<\beta\}\subseteq\varphi(\mathbb{D})$.

Proof From (3.1), it follows that

$$\frac{zf''_\xi(z)}{f'_\xi(z)}=\frac{zN''_\xi(z)}{N'_\xi(z)}+\left(\frac{1}{\xi}-1\right)\frac{zN'_\xi(z)}{N_\xi(z)}. \tag{3.4}$$

Now using the Weierstrass decompositions of the functions N_ξ and N'_ξ, which are given below

$$N_\xi(z)=\frac{Q(\xi)z^\xi}{2^\xi\Gamma(\xi+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\chi^2_{\xi,n}}\right) \text{ and } N'_\xi(z)=\frac{Q(\xi)\xi z^{\xi-1}}{2^\xi\Gamma(\xi+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\tilde{\chi}^2_{\xi,n}}\right),$$

where $\chi_{\xi,n}$ and $\tilde{\chi}_{\xi,n}$ are the nth positive roots of N_ξ and N'_ξ, we have from (3.4) that

$$1+\frac{zf''_\xi(z)}{f'_\xi(z)}=1-\left(\frac{1}{\xi}-1\right)\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n}-z^2}-\sum_{n\geq 1}\frac{2z^2}{\tilde{\chi}^2_{\xi,n}-z^2}. \tag{3.5}$$

Using the Lemma [13] in (3.5) that if $z\in\mathbb{C}$ and $\chi\in\mathbb{R}$ are such that $|z|\leq r<\chi$, then

$$\mathrm{Re}\left(\frac{z}{\chi-z}\right)\leq\left|\frac{z}{\chi-z}\right|\leq\frac{|z|}{\chi-|z|}.$$

It follows that the inequalities

$$\left|\frac{zf''_\xi(z)}{f'_\xi(z)}\right|\leq\left(\frac{1}{\xi}-1\right)\sum\frac{2r^2}{\chi^2_{\xi,n}-r^2}+\sum_{n\geq 1}\frac{2r^2}{\tilde{\chi}^2_{\xi,n}-r^2}=-\frac{rf''_\xi(r)}{f'_\xi(r)}$$

hold in $|z|=r<\tilde{\chi}_{\xi,1}$ for $\xi\leq 1$. Observe that the inequality $\left|zf''_\xi(z)/f'_\xi(z)\right|\leq -rf''_\xi(r)/f'_\xi(r)$ also holds for $\xi>1$ in $|z|=r<\tilde{\chi}_{\xi,1}$. Let $R^c_\varphi(f_\xi)$ be the smallest positive root of the equation

$$\frac{rf_\xi''(r)}{f_\xi'(r)} + \beta = 0.$$

Now let $\{w : |w-1| < \beta\} \subseteq \varphi(\mathbb{D})$ such that $\varphi(-1) = 1-\beta$. Then it follows that

$$\left|\frac{zf_\xi''(z)}{f_\xi'(z)}\right| \leq -\frac{rf_\xi''(r)}{f_\xi'(r)} \leq \beta$$

hold in $|z| = r < R_\varphi^c(f_{f_\xi})$ for $\xi \geq \max\{0, \xi_0\},\ \xi \neq 0$, which implies that the function $f_\xi \in \mathcal{C}(\varphi)$ in $|z| = r < R_\varphi^c(f_{f_\xi})$ for $\xi \geq \max\{0, \xi_0\},\ \xi \neq 0$. For the sharpness, let us consider $z = -R_\varphi^c(f_\xi)$. Then

$$\left|\frac{zf_\xi''(z)}{f_\xi'(z)}\right| = -\frac{rf_\xi''(r)}{f_\xi'(r)} = \beta$$

such that $1 + zf_\xi''(z)/f_\xi'(z) \notin \varphi(\mathbb{D})$ for all $|z| = r \geq R_\varphi^c(f_\xi)$, which proves the sharpness part. Reasoning along the same lines, proofs of the parts *b*) and *c*) follow. □

The following corollary yields the conditions on parameters for functions to be in $\mathcal{C}(\varphi)$.

Corollary 2 (Sufficient conditions for Ma–Minda convexity) *Let* $\xi \geq \max\{0, \xi_0\}$, $\xi \neq 0$. *Let* $\varphi(-1) = 1-\beta$ *and* β *is the radius of the largest disk* $\{w : |w-1| < \beta\} \subseteq \varphi(\mathbb{D})$. *Then*

(i) The function f_ξ belongs to $\mathcal{C}(\varphi)$, if

$$\left(1 - \frac{1}{\xi}\right)\frac{N_\xi'(1)}{N_\xi(1)} - \frac{N_\xi''(1)}{N_\xi'(1)} < \beta \quad \text{and} \quad 1 \geq \xi \geq \max\{0, \xi_0\},\ \xi \neq 0 \quad \text{and} \quad \xi > 1.$$

(ii) The function g_ξ belongs to $\mathcal{C}(\varphi)$, if

$$\frac{N_\xi''(1) + 2(1-\xi)N_\xi'(1) + (\xi^2+\xi)N_\xi(1)}{(1-\xi)N_\xi(1) - N_\xi'(1)} < \beta.$$

(iii) The function h_ξ belongs to $\mathcal{C}(\varphi)$, if

$$\frac{1}{2}\frac{N_\xi''(1) + (3-2\xi)N_\xi'(1) + (\xi^2-2\xi)N_\xi(1)}{(\xi-2)N_\xi(1) - N_\xi'(1)} < \beta.$$

The Fig. 2 was created by taking $\xi = 1/2,\ a = 2,\ b = 3,\ c = 0$ and $\varphi(z) = z + \sqrt{1+z^2}$ in Theorem 4.

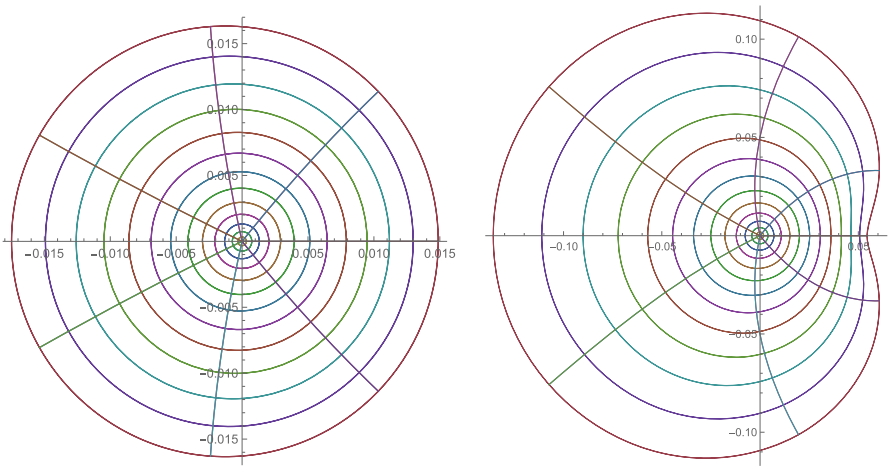


Fig. 2 Images of function $f_{1/2}(z)$ for $r = 0.1271$ and $r = 0.3$, respectively as per Theorem 4, Table 2 and Remark 1

Proposition 1 *Let β denotes the radius of the largest disk $\{w : |w-1| < \beta\}$ inside $\varphi(\mathbb{D})$. Then Theorems 3 and 4 hold for each of the following cases and the radii are sharp, where*

(i) $\beta = \min\left\{\left|1-\frac{1+A}{1+B}\right|, \left|1-\frac{1-A}{1-B}\right|\right\} = \frac{A-B}{1+|B|}$ when $\varphi(z) = \frac{1+Az}{1+Bz}$, where $-1 \le B < A \le 1$

(ii) $\beta = \sqrt{2-2\sqrt{2}+\sqrt{-2+2\sqrt{2}}}$ when $\varphi(z) = \sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$

(iii) $\beta = \sqrt{2}-1$ when $\varphi(z) = \sqrt{1+z}$

(iv) $\beta = 1-1/e$ when $\varphi(z) = e^z$

(v) $\beta = 2-\sqrt{2}$ when $\varphi(z) = z+\sqrt{1+z^2}$

(vi) $\beta = \frac{e-1}{e+1}$ when $\varphi(z) = \frac{2}{1+e^{-z}}$

(vii) $\beta = \sin 1$ when $\varphi(z) = 1+\sin z$

(viii) $\beta = 1-e^{e^{-1}-1}$ when $\varphi(z) = e^{e^z-1}$.

We now conclude with the following important observation.

Remark 3 Note that in Theorems 3 and 4, we proved that for the special functions f to be f_ξ, g_ξ and h_ξ as above

(i) $\mathcal{S}^*(\varphi) - radius = \mathcal{S}^*(1+(1-\alpha)z) - radius = \mathcal{S}^*(\alpha) - radius$

(ii) $\mathcal{C}(\varphi) - radius = \mathcal{C}(1+(1-\alpha)z) - radius = \mathcal{C}(\alpha) - radius$,

Table 2 Radii of $\mathcal{C}(\varphi)$ for the functions $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$ with $\varphi(z) = z + \sqrt{1+z^2}$

	$b=3$ and $c=0$			$a=1$ and $c=0$			$a=1$ and $b=2$		
	$a=2$	$a=3$	$a=4$	$b=2$	$b=3$	$b=4$	$c=2$	$c=3$	$c=4$
$r^c_\varphi(f_{\frac{1}{2}})$	0.1271	0.0983	0.0732	0.1443	0.1644	0.1761	0.2512	0.2812	0.3044
$r^c_\varphi(g_{\frac{1}{2}})$	0.1749	0.1352	0.1005	0.1987	0.2266	0.2427	0.3482	0.3906	0.4234
$r^c_\varphi(h_{\frac{1}{2}})$	0.0759	0.0451	0.0248	0.0983	0.1283	0.1474	0.3124	0.3979	0.4719

whenever $\varphi(-1) = 1-\alpha$ and $\beta := 1-\alpha$ is the radius of the largest disk inside $\varphi(\mathbb{D})$.

Note that the bounds obtained in [19, Section 2.2, Theorem 2.3–2.4] for the radius of starlikeness and convexity of g_ξ and h_ξ hold for $\mathcal{S}^*(\varphi)$-radius and $\mathcal{C}(\varphi)$-radius in general, however obtaining them in view of the Remark 3 is still open.

4 Radii of γ-Spirallike and convex γ-Spirallike of the functions f_ξ, g_ξ and h_ξ

In this section, we generalize Theorems A and B in view of Question 2 by proving Theorem 5 and Theorem 6 to the classes $\mathcal{S}^\gamma_p(\alpha)$ and $\mathcal{CS}^\gamma_p(\alpha)$, respectively. The first principal result we established concerns the radii of γ-Spirallike of order α and reads as follows.

Theorem 5 *Let $\alpha \in [0,1)$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The following statements hold*:

(a) If $\xi \geq \max\{0, \xi_0\}$, $\xi \neq 0$ then the radius of γ-Spirallike of order α of the function f_ξ is the smallest positive root of the equation

$$\frac{1}{\xi}\frac{ar^3 J'''_\xi(r) + (2a+b)r^2 J''_\xi(r) + (b+c)rJ'_\xi(r)}{ar^2 J''_\xi(r) + brJ'_\xi(r) + cJ_\xi(r)} = 1 - (1-\alpha)\cos\gamma.$$

(b) If $\xi \geq \max\{0, \xi_0\}$, then the radius of γ-Spirallike of order α of the function g_ξ is the smallest positive root of the equation

$$\frac{ar^3 J'''_\xi(r) + (2a+b)r^2 J''_\xi(r) + (b+c)rJ'_\xi(r)}{ar^2 J''_\xi(r) + brJ'_\xi(r) + cJ_\xi(r)} = \xi - (1-\alpha)\cos\gamma.$$

(c) If $\xi \geq \max\{0, \xi_0\}$, then the radius of γ-Spirallike of order α of the function h_ξ is the smallest positive root of the equation

$$\frac{ar\sqrt{r} J'''_\xi(\sqrt{r}) + (2a+b)rJ''_\xi(\sqrt{r}) + (b+c)\sqrt{r}J'_\xi(\sqrt{r})}{arJ''_\xi(\sqrt{r}) + b\sqrt{r}J'_\xi(\sqrt{r}) + cJ_\xi(\sqrt{r})} = \xi - 2(1-\alpha)\cos\gamma.$$

Proof Firstly, we prove part **(a)** for $\xi \geq \max\{0, \xi_0\},\ \xi \neq 0$ and **(b)** and **(c)** for $\xi \geq \max\{0, \xi_0\}$. We need to show that the following inequalities for $\alpha \in [0, 1)$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\mathrm{Re}\left(e^{-i\gamma}\frac{zf'_\xi(z)}{f_\xi(z)}\right) > \alpha\cos\gamma, \quad \mathrm{Re}\left(e^{-i\gamma}\frac{zg'_\xi(z)}{g_\xi(z)}\right) > \alpha\cos\gamma \text{ and } \mathrm{Re}\left(e^{-i\gamma}\frac{zh'_\xi(z)}{h_\xi(z)}\right) > \alpha\cos\gamma \tag{4.1}$$

are valid for $z \in \mathbb{D}_{r^*_{sp}(\alpha,\gamma;f_\xi)}$, $z \in \mathbb{D}_{r^*_{sp}(\alpha,\gamma;g_\xi)}$ and $z \in \mathbb{D}_{r^*_{sp}(\alpha,\gamma;h_\xi)}$ respectively, and each of the above inequalities does not hold in larger disks. **(a):** When we write the Eq. (1.3) in definition of the function $f_\xi(z)$, we get by using logarithmic derivation

$$\frac{zf'_\xi(z)}{f_\xi(z)} = \frac{1}{\xi}\frac{zN'_\xi(z)}{N_\xi(z)} = 1 - \frac{1}{\xi}\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n} - z^2}, \quad (\xi \geq \max\{0, \xi_0\},\ \xi \neq 0). \tag{4.2}$$

It is known [13] that if $z \in \mathbb{C}$ and $\chi \in \mathbb{R}$ are such that $|z| \leq r < \chi$, then

$$\mathrm{Re}\left(\frac{z}{\chi - z}\right) \leq \left|\frac{z}{\chi - z}\right| \leq \frac{|z|}{\chi - |z|}. \tag{4.3}$$

Then the inequality

$$\mathrm{Re}\left(\frac{z^2}{\chi^2_{\xi,n} - z^2}\right) \leq \left|\frac{z^2}{\chi^2_{\xi,n} - z^2}\right| \leq \frac{|z|^2}{\chi^2_{\xi,n} - |z|^2}$$

holds for every $|z| < \chi_{\xi,1}$. Therefore, from (4.2) and (4.3), we have

$$\begin{aligned}
\mathrm{Re}\left(e^{-i\gamma}\frac{zf'_\xi(z)}{f_\xi(z)}\right) &= \mathrm{Re}\left(e^{-i\gamma}\right) - \frac{1}{\xi}\mathrm{Re}\left(e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n} - z^2}\right) \\
&\geq \cos\gamma - \frac{1}{\xi}\left|e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n} - z^2}\right| \geq \cos\gamma - \frac{1}{\xi}\sum_{n\geq 1}\frac{2|z|^2}{\chi^2_{\xi,n} - |z|^2} \\
&= \frac{|z|f'_\xi(|z|)}{f_\xi(|z|)} + \cos\gamma - 1,
\end{aligned}$$

with equality when $z = |z| = r$. Thus, for $r \in \left(0, \chi_{\xi,1}\right)$ it follows that

$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(e^{-i\gamma}\frac{zf'_\xi(z)}{f_\xi(z)} - \alpha\cos\gamma\right)\right\} = \frac{|z|f'_\xi(|z|)}{f_\xi(|z|)} + (1-\alpha)\cos\gamma - 1.$$

Now, the mapping $\Theta_\xi : \left(0, \chi_{\xi,1}\right) \longrightarrow \mathbb{R}$ defined by

$$\Theta_\xi(r) = \frac{rf'_\xi(r)}{f_\xi(r)} + (1-\alpha)\cos\gamma - 1 = (1-\alpha)\cos\gamma - \frac{1}{\xi}\sum_{n\geq 1}\left(\frac{2r^2}{\chi^2_{\xi,n} - r^2}\right).$$

is strictly decreasing since

$$\Theta'_{\xi}(r) = -\frac{1}{\xi}\sum_{n\geq 1}\left(\frac{4r\chi^2_{\xi,n}}{\left(\chi^2_{\xi,n}-r^2\right)^2}\right) < 0$$

for all $\xi \geq \max\{0, \xi_0\}$, $\xi \neq 0$. On the other hand, since

$$\lim_{r\searrow 0}\Theta_{\xi}(r) = (1-\alpha)\cos\gamma > 0 \text{ and } \lim_{r\nearrow \chi_{\xi,1}}\Theta_{\xi}(r) = -\infty,$$

in view of the minimum principle for harmonic functions imply that the corresponding inequality in (4.1) for $\xi \geq \max\{0, \xi_0\}$, $\xi \neq 0$ holds if and only if $z \in \mathbb{D}_{r_f}$, where $r_f \in \left(0, \chi_{\xi,1}\right)$ is the smallest positive root of equation

$$\frac{rf'_{\xi}(r)}{f_{\xi}(r)} = 1-(1-\alpha)\cos\gamma,$$

which is equivalent to $\frac{rN'_{\xi}(r)}{\xi N_{\xi}(r)} = 1-(1-\alpha)\cos\gamma$.

(b): Since

$$\frac{zg'_{\xi}(z)}{g_{\xi}(z)} = (1-\xi)+\frac{zN'_{\xi}(z)}{N_{\xi}(z)} = 1-\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n}-z^2}, \quad (\xi \geq \max\{0,\xi_0\}). \tag{4.4}$$

By using the inequality (4.3), for all $z \in \mathbb{D}_{\chi_{\xi,1}}$ we get the inequality

$$\begin{aligned}\operatorname{Re}\left(e^{-i\gamma}\frac{zg'_{\xi}(z)}{g_{\xi}(z)}\right) &= \operatorname{Re}\left(e^{-i\gamma}\right) - \operatorname{Re}\left(e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n}-z^2}\right)\\ &\geq \cos\gamma - \left|e^{-i\gamma}\sum_{n\geq 1}\frac{2z^2}{\chi^2_{\xi,n}-z^2}\right| \geq \cos\gamma - \sum_{n\geq 1}\frac{2|z|^2}{\chi^2_{\xi,n}-|z|^2}\\ &= \frac{rg'_{\xi}(r)}{g_{\xi}(r)}+\cos\gamma-1,\end{aligned}$$

where $|z| = r$. Thus, for $r \in \left(0, \chi_{\xi,1}\right)$ we obtain

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(e^{-i\gamma}\frac{zg'_{\xi}(z)}{g_{\xi}(z)}-\alpha\cos\gamma\right)\right\} = \frac{rg'_{\xi}(r)}{g_{\xi}(r)}+(1-\alpha)\cos\gamma-1.$$

The function Φ_{ξ}: $\left(0, \chi_{\xi,1}\right) \longrightarrow \mathbb{R}$ defined by

$$\Phi_{\xi}(r) = \frac{rg'_{\xi}(r)}{g_{\xi}(r)}+(1-\alpha)\cos\gamma-1 = (1-\alpha)\cos\gamma-\sum_{n\geq 1}\left(\frac{2r^2}{\chi^2_{\xi,n}-r^2}\right)$$

is strictly decreasing and $\lim_{r\searrow 0}\Phi_{\xi}(r) = (1-\alpha)\cos\gamma > 0$ and $\lim_{r\nearrow\chi_{\xi,1}}\Phi_{\xi}(r) = -\infty$. Consequently, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_g}$, we have that

$$\operatorname{Re}\left(e^{-i\gamma}\frac{zg'_\xi(z)}{g_\xi(z)}\right) > \alpha\cos\gamma,$$

if and only if r_g is the smallest positive root of equation

$$\frac{rg'_\xi(r)}{g_\xi(r)} = 1-(1-\alpha)\cos\gamma \text{ or } \frac{rN'_\xi(r)}{N_\xi(r)} = \xi-(1-\alpha)\cos\gamma$$

situated in $\left(0,\chi_{\xi,1}\right)$. **(c):** Observe that

$$\frac{zh'_\xi(z)}{h_\xi(z)} = (1-\frac{\xi}{2})+\frac{1}{2}\frac{\sqrt{z}N'_\xi(\sqrt{z})}{N_\xi(\sqrt{z})} = 1-\sum_{n\geq 1}\frac{z}{\chi^2_{\xi,n}-z}, \quad (\xi\geq\max\{0,\xi_0\}). \tag{4.5}$$

By using the inequality (4.2), for all $z\in\mathbb{D}_{\chi^2_{\xi,1}}$ we obtain

$$\begin{aligned}\operatorname{Re}\left(e^{-i\gamma}\frac{zh'_\xi(z)}{h_\xi(z)}\right) &= \operatorname{Re}\left(e^{-i\gamma}\right)-\operatorname{Re}\left(e^{-i\gamma}\sum_{n\geq 1}\frac{z}{\chi^2_{\xi,n}-z}\right)\\ &\geq \cos\gamma-\left|e^{-i\gamma}\sum_{n\geq 1}\frac{z}{\chi^2_{\xi,n}-z}\right| \geq \cos\gamma-\sum_{n\geq 1}\frac{2|z|}{\chi^2_{\xi,n}-|z|}\\ &= \frac{rh'_\xi(r)}{h_\xi(r)}+\cos\gamma-1,\end{aligned}$$

where equality holds for $z=|z|=r$. As a result, for $r\in\left(0,\chi_{\xi,1}\right)$ it follows that

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(e^{-i\gamma}\frac{zh'_\xi(z)}{h_\xi(z)}-\alpha\cos\gamma\right)\right\} = \frac{rh'_\xi(r)}{h_\xi(r)}+(1-\alpha)\cos\gamma-1.$$

The function $\Psi_\xi\colon\left(0,\chi^2_{\xi,1}\right)\longrightarrow\mathbb{R}$ defined by

$$\Psi_\xi(r) = \frac{rh'_\xi(r)}{h_\xi(r)}+(1-\alpha)\cos\gamma-1 = (1-\alpha)\cos\gamma-\sum_{n\geq 1}\left(\frac{2r}{\chi^2_{\xi,n}-r}\right).$$

is strictly decreasing, since

$$\Psi'_\xi(r) = -\sum_{n\geq 1}\left(\frac{\chi^2_{\xi,n}}{\left(\chi^2_{\xi,n}-r\right)^2}\right)<0$$

for all $\xi\geq\max\{0,\xi_0\}$. On the other hand, since

$$\lim_{r\searrow 0}\Psi_\xi(r) = (1-\alpha)\cos\gamma>0 \text{ and } \lim_{r\nearrow\chi^2_{\xi,1}}\Psi_\xi(r) = -\infty$$

in view of the minimum principle for harmonic functions imply that the corresponding inequality in (4.1) holds if and only if $z\in\mathbb{D}_{r_h}$, where r_h is the smallest positive root of equation

$$\frac{rh'_{\xi}(r)}{h_{\xi}(r)} = 1 - (1-\alpha)\cos\gamma,$$

which is equivalent to

$$\frac{\sqrt{r}N'_{\xi}(\sqrt{r})}{N_{\xi}(\sqrt{r})} = 1 - (1-\alpha)\cos\gamma$$

and r_h situated in $\left(0, \chi^2_{\xi,1}\right)$. This completes the proof of part **(a)** when $\xi \geq \max\{0, \xi_0\}$, $\xi \neq 0$, and parts **(b)** and **(c)** when $\xi \geq \max\{0, \xi_0\}$. □

Remark 4 In Theorem 5, the choice of $\gamma = 0$ yields the result [19, Theorem 2.1] for the class of starlike functions of order α.

Taking $\xi = 1/2$, $a = 1$, $b = 2$, $c = 0$, $\alpha = 1/2$ and $\gamma = \pi/3$ in Theorem 5, we have following results.

Corollary 3 *The following statements are true.*

(i) The radius of $\frac{\pi}{3}$-Spirallike of order 1/2 of the function $f_{1/2}$ is the smallest positive root of the equation

$$\frac{4x(8x+1)\cos x + (68x^2-7)\sin x}{4(-4x\cos x + (4x^2+1)\sin x)} = 0.$$

(ii) The radius of $\frac{\pi}{3}$-Spirallike of order 1/2 of the function $g_{1/2}$ is the smallest positive root of the equation

$$\frac{16x^3\cos x + 3(12x^2-1)\sin x}{4(-4x\cos x + (4x^2+1)\sin x)} = 0.$$

(iii) The radius of $\frac{\pi}{3}$-Spirallike of order 1/2 of the function $h_{1/2}$ is the smallest positive root of the equation

$$\frac{2\sqrt{x}(4x-1)\cos\sqrt{x} + (20x-1)\sin\sqrt{x}}{2(-4\sqrt{x}\cos\sqrt{x} + (4x+1)\sin\sqrt{x})} = 0.$$

The Fig. 3 was created by taking the special values in the Corollary 3.

Remark 5 When $\xi = 0.5$, considering the special values of a, b and c real numbers, radii of γ-Spirallike of order α of the function f_{ξ} is seen from the table above. In addition, according to the increasing values of a, b and c reel numbers, it is clear that radii of γ-Spirallike of order α of the function f_{ξ} is monotone.

In the following, we deal with convex analogue of the class of γ-Spiralllike functions of order α. Hence, we generalize Theorem B.

Theorem 6 *Let* $\alpha \in [0, 1)$ *and* $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *The following statements hold*:

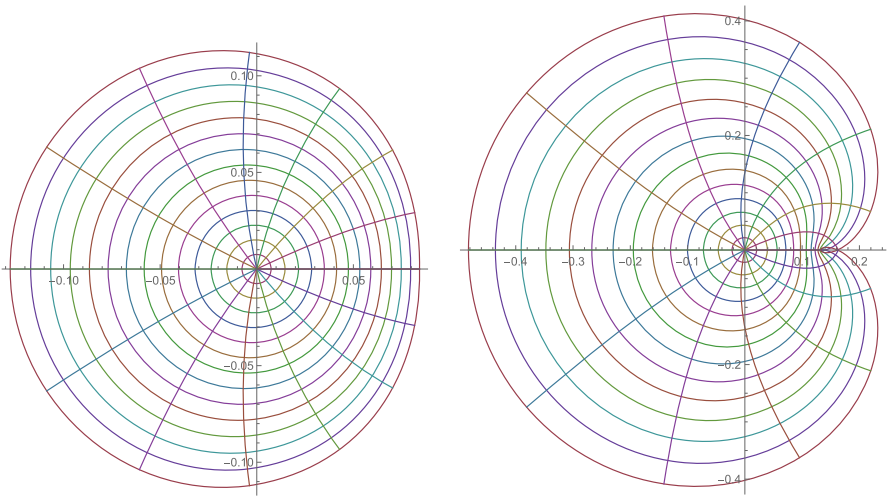


Fig. 3 Images of function $h_{1/2}(z)$ for $r = 0.1056$ and $r = 0.3$, respectively as per Corollary 3 and Table 3

(a) If $\xi \geq \max\{0, \xi_0\}$, $\xi \neq 0$ then the radius of the convex γ-Spirallike of order α of the function f_ξ is the smallest positive root of the equation

$$\frac{rN_\xi''(r)}{N_\xi'(r)} + \left(\frac{1}{\xi} - 1\right)\frac{rN_\xi'(r)}{N_\xi(r)} = (\alpha - 1)\cos\gamma.$$

(b) If $\xi \geq \max\{0, \xi_0\}$, then the radius of the convex γ-Spirallike of order α of the function g_ξ is the smallest positive root of the equation

$$\frac{r^2N_\xi''(r) + (2 - 2\xi)rN_\xi'(r) + \left(\xi^2 - \xi\right)N_\xi(r)}{rN_\xi'(r) + (1 - \xi)N_\xi(r)} = (\alpha - 1)\cos\gamma.$$

(c) If $\xi \geq \max\{0, \xi_0\}$, then the radius of the convex γ-Spirallike of order α of the function h_ξ is the smallest positive root of the equation

$$\frac{rN_\xi''(\sqrt{r}) + (3 - 2\xi)\sqrt{r}N_\xi'(\sqrt{r}) + \left(\xi^2 - 2\xi\right)N_\xi(\sqrt{r})}{2\sqrt{r}N_\xi'(\sqrt{r}) + 2(2 - \xi)N_\xi(\sqrt{r})} = (\alpha - 1)\cos\gamma.$$

Proof We first prove the part (a). From (1.3), we have

$$1 + \frac{zf_\xi''(z)}{f_\xi'(z)} = 1 + \frac{zN_\xi''(z)}{N_\xi'(z)} + \left(\frac{1}{\xi} - 1\right)\frac{zN_\xi'(z)}{N_\xi(z)}$$

and by means of (1.6) and (1.7), we obtain

$$1+\frac{zf_\xi''(z)}{f_\xi'(z)}=1-\left(\frac{1}{\xi}-1\right)\sum_{n\geq 1}\frac{2z^2}{\chi_{\xi,n}^2-z^2}-\sum_{n\geq 1}\frac{2z^2}{\chi_{\xi,n}'^2-z^2}.$$

For $1\geq\xi>\max\{0,\xi_0\}$ by using (4.3), we get

$$\begin{aligned}
&\mathrm{Re}\left(e^{-i\gamma}\left(1+\frac{zf_\xi''(z)}{f_\xi'(z)}\right)\right)\\
&=\mathrm{Re}(e^{-i\gamma})-\mathrm{Re}\left(e^{-i\gamma}\left(\sum_{n\geq 1}\frac{2z^2}{\chi_{\xi,n}'^2-z^2}+\left(\frac{1}{\xi}-1\right)\sum_{n\geq 1}\frac{2z^2}{\chi_{\xi,n}^2-z^2}\right)\right)\\
&\geq\cos\gamma-\left(\frac{1}{\xi}-1\right)\sum_{n\geq 1}\frac{2r^2}{\chi_{\xi,n}^2-r^2}-\sum_{n\geq 1}\frac{2r^2}{\chi_{\xi,n}'^2-r^2}\\
&\geq\cos\gamma+\frac{rf_\xi''(r)}{f_\xi'(r)},
\end{aligned}\tag{4.6}$$

where $|z|=r$. Moreover, observe that if we use the inequality [4, Lemma 2.1]

$$\mu\mathrm{Re}\left(\frac{z}{a-z}\right)-\mathrm{Re}\left(\frac{z}{b-z}\right)\geq\mu\frac{|z|}{a-|z|}-\frac{|z|}{b-|z|}$$

where $a>b>0$, $\mu\in[0,1]$ and $z\in\mathbb{C}$ such that $|z|<b$, then we get that the inequality (4.6) is also valid when $\xi\geq 1$. Here we used that the zeros of N_ξ and N_ξ' are interlacing according to Lemma 1. The above inequality implies for $r\in(0,\chi_{\xi,1}')$

$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(e^{-i\gamma}\left(1+\frac{zf_\xi''(z)}{f_\xi'(z)}\right)-\alpha\cos\gamma\right)\right\}=(1-\alpha)\cos\gamma+\frac{rf_\xi''(r)}{f_\xi'(r)}.$$

Now, the proof of part (a) follows on similar lines as of Theorem 5.

For the other parts, note that the functions g_ξ and h_ξ belong to the Laguerre-Pólya class $\mathcal{LP}$, which is closed under differentiation, their derivatives g_ξ' and h_ξ' also belong to $\mathcal{LP}$ and the zeros are real. Thus assuming $\delta_{\xi,n}$ and $\gamma_{\xi,n}$ are the positive zeros of g_ξ' and h_ξ', respectively, we have the following representations:

$$g_\xi'(z)=\prod_{n\geq 1}\left(1-\frac{z^2}{\delta_{\xi,n}^2}\right)\quad\text{and}\quad h_\xi'(z)=\prod_{n\geq 1}\left(1-\frac{z}{\gamma_{\xi,n}}\right),$$

which yield

$$1+\frac{zg_\xi''(z)}{g_\xi'(z)}=1-\sum_{n\geq 1}\frac{2z^2}{\delta_{\xi,n}^2-z^2}\quad\text{and}\quad 1+\frac{zh_\xi''(z)}{h_\xi'(z)}=1-\sum_{n\geq 1}\frac{z}{\gamma_{\xi,n}-z}.$$

Further, reasoning along the same lines as in Theorem 5, the result follows at once. □

Remark 6 In Theorem 6, the choice of $\gamma=0$ yields the result [19, Theorem 2.2] for the class of convex functions of order α (Table 4).

Table 3 Radii of $\frac{\pi}{3}$-Spirallike of order 1/2 for $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$

	$b=3$ and $c=0$			$a=1$ and $c=0$			$a=1$ and $b=2$		
	$a=2$	$a=3$	$a=4$	$b=2$	$b=3$	$b=4$	$c=2$	$c=3$	$c=4$
$R\left(f_{\frac{1}{2}}\right)$	0.1539	0.1190	0.0886	0.1746	0.1990	0.2130	0.3038	0.3401	0.3680
$R\left(g_{\frac{1}{2}}\right)$	0.2121	0.1639	0.1220	0.2409	0.2747	0.2942	0.4217	0.4730	0.5126
$R\left(h_{\frac{1}{2}}\right)$	0.0817	0.0486	0.0268	0.1056	0.1377	0.1582	0.3309	0.4193	0.4953

Table 4 Radii of the convex $\frac{\pi}{3}$-Spirallike of order 1/2 for $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$

	$b=3$ and $c=0$			$a=1$ and $c=0$			$a=1$ and $b=2$		
	$a=2$	$a=3$	$a=4$	$b=2$	$b=3$	$b=4$	$c=2$	$c=3$	$c=4$
$R^c\left(f_{\frac{1}{2}}\right)$	0.0875	0.0677	0.0504	0.0993	0.1131	0.1210	0.1722	0.1925	0.2082
$R^c\left(g_{\frac{1}{2}}\right)$	0.1221	0.0944	0.0703	0.1386	0.1579	0.1691	0.2411	0.2699	0.2921
$R^c\left(h_{\frac{1}{2}}\right)$	0.0406	0.0242	0.0134	0.0524	0.0682	0.0783	0.1622	0.2046	0.2408

Authors' contributions All authors contributed equally.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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