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Sharp Coefficients Bounds for Starlike Functions Associated with Gregory Coefficients

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Abstract
In this paper we introduced the class $\mathcal{S}_G^*$ of analytic functions which is related with starlike functions and generating function of Gregory coefficients. By using bounds on some coefficient functionals for the family of functions with positive real part, we obtain several sharp coefficient bounds on the first six coefficients and also further bounds on the corresponding Hankel determinants for functions in the class $\mathcal{S}_G^*$. Additionally, the sharp bounds for logarithmic and inverse coefficients of functions belonging to the considered class $\mathcal{S}_G^*$ were estimated.

Keywords Univalent functions · Starlike function · Gregory coefficients · Hankel determinant · Logarithmic coefficients · Inverse coefficients

Mathematics Subject Classification Primiary 30C45; Secondary 30C50 · 30C80

1 Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let us denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ containing functions which are univalent in $\mathcal{U}$. An analytic function f is subordinate to an analytic function g (written as $f \prec g$) if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathcal{U}$, then $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

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In 1992, Ma and Minda [31] gave a unified presentation of various subclasses of starlike and convex functions by replacing the subordinate function $(1+z)/(1-z)$ by a more general analytic function φ with positive real part and normalized by the conditions $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps $\mathcal{U}$ onto univalently a starlike region with respect to 1 and symmetric with respect to the real axis. They introduced the following general class that envelopes several well-known classes as special cases: $\mathcal{S}^*[\varphi] = \left\{f \in \mathcal{A} : zf'(z)/f(z) \prec \varphi(z)\right\}$. In literature, the functions belonging to this class is called Ma-Minda starlike functions. For $-1 \leq B < A \leq 1$, $\mathcal{S}^*[(1+Az)/(1+Bz)] := \mathcal{S}^*[A,B]$ are called the Janowski starlike functions, introduced by Janowski [20]. The class $\mathcal{S}^*(\beta)$ of starlike functions of order β $(0 \leq \beta < 1)$ defined by taking $\varphi(z) = (1+(1-2\beta)z)/(1-z)$. Note that $\mathcal{S}^* = \mathcal{S}^*(0)$ is the classical class of starlike functions. By taking $\varphi(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\left(1+\sqrt{z}\right)/\left(1-\sqrt{z}\right)\right)\right)^2$, we obtain the class $\mathcal{S}_p$ of parabolic starlike functions, introduced by Rønning [40]. $\mathcal{S}^*[\beta,-\beta] := \widetilde{\mathcal{S}^*}(\beta) = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta|zf'(z)/f(z) + 1|\}$ has been studied in [1, 2].

Recently, the coefficient problem for $\mathcal{S}^*[\varphi]$ in the case of generated functions of some well-known numbers of φ studied by [9, 14, 15, 26, 38]. For example, the case when φ is defined by

$$\varphi(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \quad (z \in \mathbb{C},\ \tau = (1-\sqrt{5})/2),$$

which generated function of Fibonacci numbers, coefficient problem studied by Dziok and co-authors [14, 15]. Similarly, the function

$$\varphi(z) = e^{e^z-1} \quad (z \in \mathbb{C})$$

is generated function of Bell numbers. The coefficient problem for this function studied by Kumar and co-authors [26]. Recently, Mendiratta et al. [32] and, Goel and Kumar [19] obtained the structural formula, inclusion relations, coefficient estimates, growth and distortion results, subordination theorems and various radii constants for the exponential function

$$\varphi(z) = e^z \quad (z \in \mathbb{C})$$

and the Sigmoid function

$$\varphi(z) = \frac{2}{1+e^{-z}} \quad (z \in \mathbb{C})$$

respectively. In 2021, Deniz [13] has studied sharp coefficient problem for the function

$$\varphi(z) = e^{z+\frac{\lambda}{2}z^2} \quad (z \in \mathbb{C}, \lambda \geq 1),$$

which is generated function of generalized telephone numbers.

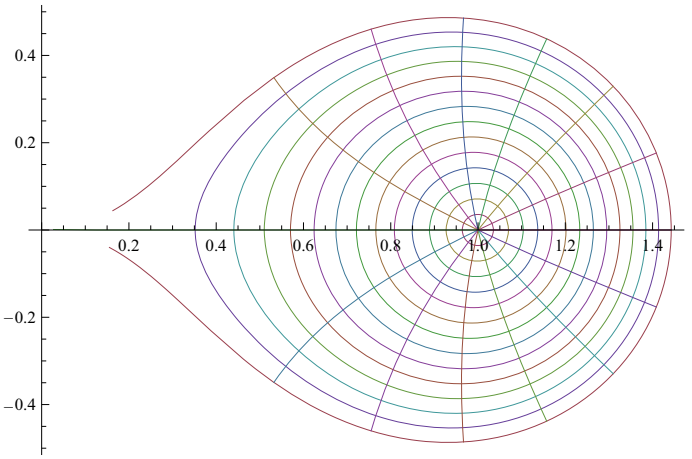


Fig. 1 Image of the open unit disk $\mathcal{U}$ under the mapping $\Psi(z)$

Motivated by the above-cited works, we consider the function φ for which $\varphi(\mathcal{U})$ is starlike with respect to 1 and whose coefficients is the Gregory coefficients. Gregory coefficients are decreasing rational numbers $1/2$, $1/12$, $1/24$, $19/720$, ..., which are similar in their role to the Bernoulli numbers and appear in a large number of problems, especially in those related to the numerical analysis and to the number theory. They first appeared in the works of the Scottish mathematician James Gregory in 1671 and were subsequently rediscovered many times. Among the famous mathematicians who rediscovered them, we find Laplace, Mascheroni, Fontana, Bessel, Clausen, Hermite, Pearson and Fisher. Furthermore, Gregory's coefficients can rightly be considered one of the most frequently rediscovered in mathematics (the last rediscovery dates back to our century), the reason for which in literature we can find them under various names (e.g. reciprocal logarithmic numbers, Bernoulli numbers of the second kind, Cauchy numbers, etc.). For the same reason, their authorship is often attributed to various mathematicians. In this paper, authors considered the generating function of the Gregory coefficients G_n (see [6, 36]) as follows:

$$\frac{z}{\ln(1+z)} = \sum_{n=0}^{\infty} G_n z^n \quad (|z| < 1).$$

Clearly, G_n for some values of n as $G_0 = 1$, $G_1 = \frac{1}{2}$, $G_2 = -\frac{1}{12}$, $G_3 = \frac{1}{24}$, $G_4 = -\frac{19}{720}$, $G_5 = \frac{3}{160}$, and $G_6 = -\frac{863}{60480}$. We now consider the function $\Psi(z) := \frac{z}{\ln(1+z)}$ with its domain of definition as the open unit disk $\mathcal{U}$. In connection with this function, we define the class (Fig. 1)

$$\mathcal{S}_G^* := \left\{ f : \ f \in \mathcal{A} \text{ and } zf'(z)/f(z) \prec \Psi(z) \right\}.$$

2 Preliminary Results

The following lemmas is needed for the main results.

Lemma 2.1 [29] *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ $(p_1 \geq 0)$, *then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{2.1}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)\left(1 - |x|^2\right) y, \tag{2.2}$$

for some $x, y \in \mathbb{C}$ *with* $|x| \leq 1$ *and* $|y| \leq 1$.

Lemma 2.2 *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ $(p_1 \geq 0)$, *then*

$$|p_n| \leq 2 \quad (n \geq 1), \tag{2.3}$$

and if $Q \in [0, 1]$ *and* $Q(2Q - 1) \leq R \leq Q$, *then*

$$\left| p_3 - 2Q p_1 p_2 + R p_1^3 \right| \leq 2. \tag{2.4}$$

Also

$$\begin{aligned} |p_{n+k} - \mu p_n p_k| &\leq 2 \max\{1, |2\mu - 1|\} \\ &= 2 \begin{cases} 1, & \text{for } 0 \leq \mu \leq 1, \\ |2\mu - 1|, & \text{otherwise} \end{cases}. \end{aligned} \tag{2.5}$$

The inequalities (2.3), (2.4) and (2.5) are taken from [8, 29] and [37], respectively.

Lemma 2.3 (see [38]) *Let* τ, ψ, ρ *and* ς *satify the inequalities* $0 < \tau < 1$, $0 < \varsigma < 1$ *and*

$$\begin{aligned} &8\varsigma(1 - \varsigma)\left[(\tau\psi - 2\rho)^2 + (\tau(\varsigma + \tau) - \psi)^2\right] + \tau(1 - \tau)(\psi - 2\varsigma\tau)^2 \\ &\quad \leq 4\varsigma\tau^2(1 - \tau)^2(1 - \varsigma). \end{aligned} \tag{2.6}$$

If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ $(p_1 \geq 0)$, *then*

$$\left| \rho p_1^4 + \varsigma p_2^2 + 2\tau p_1 p_3 - \frac{3}{2}\psi p_1^2 p_2 - p_4 \right| \leq 2. \tag{2.7}$$

Lemma 2.4 [35] *Let* $\overline{\mathcal{U}} = \{z : |z| \leq 1\}$. *Also, for any real numbers* a, b *and* c, *let the quantity* $Y(a, b, c) = \max_{z \in \overline{\mathcal{U}}} \left\{ \left| a + bz + cz^2 \right| + 1 - |z|^2 \right\}$. *If* $ac \geq 0$, *then*

$$Y(a, b, c) = \begin{cases} |a| + |b| + |c| & (|b| \geq 2(1 - |c|)) \\ 1 + |a| + \frac{b^2}{4(1 - |c|)} & (|b| < 2(1 - |c|)) \end{cases}.$$

Furthermore, if $ac < 0$, *then*

$$Y(a,b,c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1-|c|)} & \left(-4ac(c^{-2}-1) \le b^2;\ |b| < 2(1-|c|)\right) \\ 1 + |a| + \frac{b^2}{4(1+|c|)} & \left(b^2 < \min\left\{4(1+|c|)^2, -4ac(c^{-2}-1)\right\}\right) \\ R(a,b,c) & (otherwise) \end{cases},$$

where

$$R(a,b,c) = \begin{cases} |a| + |b| - |c| & (|c|\,(|b| + 4\,|a|) \le |ab|)) \\ -|a| + |b| + |c| & (|ab| \le |c|\,(|b| - 4\,|a|)) \\ (|a| + |c|)\sqrt{1 - \frac{b^2}{4ac}} & (otherwise) \end{cases}.$$

3 Coefficient Estimates

Finding the upper bound for coefficients have been one of the central topic of research in geometric function theory as it gives several properties of functions. Therefore, we will be interested in the following problem in this section.

Problem: Find $\sup |a_n|$ $(n = 2, 3, \ldots)$ for certain sublasses of univalent functions. In particular, bound for the second coefficient gives growth and distortion theorems for functions in the class. Another one is the coefficient problem related with Hankel determinants. Similarly, using the Hankel determinants (which also deals with the bound on coefficients), Cantor [7] proved that "if ratio of two bounded analytic functions in $\mathcal{U}$, then the function is rational". The Hankel determinants [33] $H_q(n)$ $(n = 1, 2, \ldots,\ q = 1, 2, \ldots)$ of the function f are defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

This determinant was discussed by several authors with $q = 2$. For example, we know that the functional $H_2(1) := a_3 - a_2^2$ is known as the Fekete-Szegö functional and they consider the further generalized functional $a_3 - \mu a_2^2$, where μ is some real number. Estimating for the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö problem. In 1969, Keogh and Merkes [24] solved the Fekete-Szegö problem for the class $\mathcal{S}^*$. The second Hankel determinant $H_2(2)$ is given by $H_2(2) := \left|a_2 a_4 - a_3^2\right|$. The bounds for the second Hankel determinant $H_2(2)$ obtained for the class $\mathcal{S}^*$ in [21]. Lee et al. [28] established the sharp bound to $|H_2(2)|$ by generalizing their classes using subordination. Moreover, the quantity given by $H_3(1) := a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$ is called third Hankel determinant. Zaprawa [48] proved that $|H_3(1)| < 1$ for $f \in \mathcal{S}^*$. In 2019, Kwon, Lecko and Sim [27] improved the result of Zaprawa as $|H_3(1)| < 8/9$. In 2022, Kowalczyk, Lecko and Thomas [25] proved the inequality $|H_3(1)| < 4/9$ for starlike functions. This result is sharp for the class

$\mathcal{S}^*$. Recently, Hankel determinants and Fekete-Szegö problem have been considered in many papers of Srivastava and his co-authors (see, for example, [41–45]) and Deniz and his co-authors (see, [12, 23]).

In this paper, we seek sharp upper bounds for the coefficients a_j $(j = 2, 3, 4, 5, 6)$ and functionals $H_2(2)$ and $H_3(1)$ for functions f belonging to the class $\mathcal{S}_G^*$.

Theorem 3.1 *Let* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_G^*$. *Then,*

$$|a_n| \leq \frac{1}{2(n-1)} \quad (n = 2, 3, 4, 5)$$
$$|a_6| \leq \frac{13}{48}.$$

All of the above estimates, except that on a_6, *are sharp.*

Proof Since $f \in \mathcal{S}_G^*$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$ such that

$$\frac{zf'(z)}{f(z)} = \Psi(w(z)) = \frac{w(z)}{\ln(1 + w(z))} \quad (z \in \mathcal{U}). \tag{3.1}$$

Define the function p by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1z + p_2z^2 + \cdots \quad (z \in \mathcal{U})$$

and thus,

$$\begin{aligned} w(z) = \frac{p(z)-1}{p(z)+1} &= \frac{p_1}{2}z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \frac{1}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)z^3 \\ &+ \frac{1}{2}\left(p_4 - p_1p_3 + \frac{3p_1^2p_2}{4} - \frac{p_2^2}{2} - \frac{p_1^4}{8}\right)z^4 \\ &+ \frac{1}{2}\left(p_5 - \frac{1}{2}p_1^3p_2 + \frac{3}{4}p_1p_2^2 + \frac{3}{4}p_1^2p_3 - p_2p_3 - p_1p_4 + \frac{1}{16}p_1^5\right)z^5 + \cdots \end{aligned} \tag{3.2}$$

in $\mathcal{U}$. Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and has positive real part in $\mathcal{U}$. By using (3.2) together with $\frac{w(z)}{\ln(1+w(z))}$, it is evident that

$$\begin{aligned} \Psi(w(z)) = 1 &+ \frac{p_1z}{4} + \frac{1}{48}\left(-7p_1^2 + 12p_2\right)z^2 + \frac{1}{192}\left(17p_1^3 - 56p_1p_2 + 48p_3\right)z^3 \\ &+ \frac{1}{11520}\left(-649p_1^4 + 3060p_1^2p_2 - 3360p_1p_3 - 1680p_2^2 + 2880p_4\right)z^4 \\ &+ \frac{1}{46080}\left(\begin{array}{l} 1739p_1^5 - 10384p_1^3p_2 + 12240p_1^2p_3 + 12240p_1p_2^2 \\ -13440p_2p_3 + 13440p_1p_4 + 11520p_5 \end{array}\right)z^5 + \cdots. \end{aligned} \tag{3.3}$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(-a_2^2 + 2a_3\right) z^2 + \left(a_2^3 - 3a_2a_3 + 3a_4\right) z^3 + \left(-a_2^4 + 4a_2^2a_3 - 2a_3^2 - 4a_2a_4 + 4a_5\right) z^4 + \left(a_2^5 - 5a_2^3a_3 + 5a_2a_3^2 + 5a_2^2a_4 - 5a_3a_4 - 5a_2a_5 + 5a_6\right) z^5 + \cdots, \tag{3.4}$$

it follows by (3.1), (3.3) and (3.4) that

$$a_2 = \frac{p_1}{4}, \tag{3.5}$$

$$a_3 = \frac{1}{24}\left(-p_1^2 + 3p_2\right), \tag{3.6}$$

$$a_4 = \frac{1}{288}\left(4p_1^3 - 19p_1p_2 + 24p_3\right), \tag{3.7}$$

$$a_5 = -\frac{1}{16}\left(\frac{71}{720}p_1^4 + \frac{11}{24}p_2^2 + \frac{5}{6}p_1p_3 - \frac{85}{144}p_1^2p_2 - p_4\right), \tag{3.8}$$

$$a_6 = \frac{\left(\begin{array}{l}2267p_1^5 - 15677p_1^3p_2 + 21720p_1^2p_3 + 23370p_2^2p_1\\ -29520p_1p_4 - 33120p_2p_3 + 34560p_5\end{array}\right)}{691200}. \tag{3.9}$$

Thus, we get

$$|a_2| \le \frac{1}{2} \quad (\text{from } (2.3)),$$

$$|a_3| = \frac{1}{8}\left|p_2 - \frac{1}{3}p_1^2\right| \le \frac{1}{4} \quad (\text{from } (2.5)),$$

$$|a_4| = \frac{1}{12}\left|p_3 - \frac{19}{24}p_1p_2 + \frac{1}{6}p_1^3\right| \le \frac{1}{6} \quad (\text{from } (2.4)),$$

$$|a_5| = \frac{1}{16}\left|\frac{71}{720}p_1^4 + \frac{11}{24}p_2^2 + \frac{5}{6}p_1p_3 - \frac{85}{144}p_1^2p_2 - p_4\right| \le \frac{1}{8} \quad (\text{from } (2.7)).$$

We now find the estimates on $|a_6|$. Therefore, from the Lemma 2.3, $|p_n| \le 2$ and (2.5), we find that

$$\begin{aligned}|a_6| &= \frac{\left|\begin{array}{l}2267p_1^5 - 15677p_1^3p_2 + 21720p_1^2p_3 + 23370p_2^2p_1\\ -29520p_1p_4 - 33120p_2p_3 + 34560p_5\end{array}\right|}{691200}\\ &\le \frac{41\,|p_1|}{960}\left|\frac{2267}{29520}p_1^4 + \frac{779}{984}p_2^2 + \frac{181}{246}p_1p_3 - \frac{15677}{29520}p_1^2p_2 - p_4\right| + \frac{1}{20}\left|p_5 - \frac{23}{24}p_2p_3\right|\\ &\le \frac{41}{240} + \frac{1}{10} = \frac{13}{48}.\end{aligned}$$

The first five results are sharp for the function $f : \mathcal{U} \to \mathbb{C}$ given by

$$f_1(z) = z \exp\left(\int_0^z \frac{\Psi(t)-1}{t}dt\right) = z + \frac{1}{2}z^2 + \cdots, \tag{3.10}$$

$$f_2(z) = z \exp\left(\int_0^z \frac{\Psi(t^2)-1}{t}dt\right) = z + \frac{1}{4}z^3 + \cdots, \tag{3.11}$$

$$f_3(z) = z \exp\left(\int_0^z \frac{\Psi(t^3)-1}{t}dt\right) = z + \frac{1}{6}z^4 + \cdots, \tag{3.12}$$

$$f_4(z) = z \exp\left(\int_0^z \frac{\Psi(t^4)-1}{t}dt\right) = z + \frac{1}{8}z^5 + \cdots, \tag{3.13}$$

$$f_5(z) = z \exp\left(\int_0^z \frac{\Psi(t^5)-1}{t}dt\right) = z + \frac{1}{10}z^6 + \cdots. \tag{3.14}$$

This completes the proof of Theorem 3.1. □

Conjecture 3.1 *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_G^*$. As we see from the extremal functions $f_i(z)$, $i = 1, 2, 3, 4$, the first five coeffitients are obtained as our sharp bound estimates for $|a_n|$, $n = 2, 3, 4, 5$. By comparing with the extremal function $f_5(z)$, the sixth coefficient is expected as the sharp bound estimate for $|a_6|$. So it is an open question whether the bound for $|a_6|$ is sharp and if one can show the following estimate $|a_6| \leq \frac{1}{10}$. Also, it is an open question the sharp inequality $|a_n| \leq \frac{1}{2(n-1)}$ for all $n \in \mathbb{N}\backslash\{1\}$.*

In 2021, Deniz [13] defined a subclass of starlike functions which is related with generalized telephone numbers. He obtained some sharp bounds of coefficients for this class. These results are as follows: $|a_2| \leq 1$, $|a_3| \leq 1$, $|a_4| \leq \frac{8}{9}$, $|a_5| \leq \frac{107}{144}$ and $|a_6| \leq \frac{2381}{3600}$. When we compare these results with Theorem 3.1, seen that our results are better.

Theorem 3.2 *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_G^*$. Then, the following sharp estimates holds:*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{4}\max\left\{1, \left|\mu - \frac{1}{3}\right|\right\}, \quad (\mu \in \mathbb{C})$$

$$|a_2a_3 - a_4| \leq \frac{1}{6}$$

and

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{16}.$$

Proof From (3.5) and (3.6), we have

$$a_3 - \mu a_2^2 = \frac{1}{24}\left(-p_1^2 + 3p_2\right) - \mu\frac{p_1^2}{16} = \frac{1}{8}\left(p_2 - \left(\frac{3\mu+2}{6}\right)p_1^2\right).$$

If we take $\nu = \frac{3\mu+2}{6}$ in known result (see [24]): $\left|p_2 - \nu p_1^2\right| \leq 2\max\{1, |2\nu - 1|\}$ for $\mu, \nu \in \mathbb{C}$, we obtain that

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{4}\max\{1, \left|\mu - \frac{1}{3}\right|\}.$$

Similarly from (3.5), (3.6) and (3.7), we have

$$a_2a_3 - a_4 = -\frac{1}{288}\left(24p_3 - 28p_1p_2 + 7p_1^3\right)$$

and so from (2.4)

$$|a_2a_3 - a_4| \leq \frac{1}{288}\left|24p_3 - 28p_1p_2 + 7p_1^3\right| = \frac{1}{12}\left|p_3 - \frac{7}{6}p_1p_2 + \frac{7}{24}p_1^3\right| \leq \frac{1}{6}.$$

Now, we investigate last estimate in Theorem 3.2. From (3.5), (3.6) and (3.7) again, we see that

$$a_2a_4 - a_3^2 = \frac{2p_1^4 - 7p_1^2p_2 - 18p_2^2 + 24p_1p_3}{1152} := T,$$

which, upon applying Lemma 2.1 and assuming that $s = p_1 \in [0, 2]$, we can write

$$T = \frac{4 - s^2}{2304}\left(-s^2x - 3\left(s^2 + 12\right)x^2 + 24s\left(1 - |x|^2\right)y\right).$$

If $s = 0$, then $T = -\frac{1}{16}x^2$. Thus, since $|x| \leq 1$, we have

$$|T| \leq \frac{1}{16}. \tag{3.15}$$

If $s = 2$, then

$$|T| = 0. \tag{3.16}$$

We now let $s \in (0, 2)$. Then, we can write

$$\begin{aligned} |T| &= \left|\frac{4 - s^2}{2304}\left(-s^2x - 3\left(s^2 + 12\right)x^2 + 24s\left(1 - |x|^2\right)y\right)\right| \\ &\leq \frac{s\left(4 - s^2\right)}{96}\left[\left|-\frac{s}{24}x - \frac{s^2 + 12}{8s}x^2\right| + 1 - |x|^2\right] \\ &= \frac{s\left(4 - s^2\right)}{96}\left[\left|\widetilde{a} + \widetilde{b}x + \widetilde{c}x^2\right| + 1 - |x|^2\right] \end{aligned}$$

where

$$\widetilde{a} = 0,\ \widetilde{b} = -\frac{s}{24},\ \widetilde{c} = -\frac{s^2+12}{8s}.$$

It follows that $\widetilde{a}\widetilde{c} = 0$. Also, we easily see that

$$\left|\widetilde{b}\right| - 2(1 - |\widetilde{c}|) = \frac{7s^2 - 48s + 72}{24s} > 0.$$

Therefore, we have

$$\begin{aligned} |T| &\leq \frac{s\left(4-s^2\right)}{96}\left(|\widetilde{a}| + \left|\widetilde{b}\right| + |\widetilde{c}|\right) = \frac{s\left(4-s^2\right)}{96}\left(\frac{s^2+9}{6s}\right) \\ &= \frac{-s^4 - 5s^2 + 36}{576}. \end{aligned}$$

Let $t = s^2$ $(t \in (0,4))$. Then, we investigate maximum of the function Φ_0 defined by

$$\Phi_0(t) = \frac{-t^2 - 5t + 36}{576}.$$

In that case, we have

$$|T| \leq |\Phi_0(t)| \leq \frac{-t^2 - 5t + 36}{576} \leq \frac{1}{16}. \tag{3.17}$$

□

Remark 3.3 By using Theorem 3.1 and Theorem 3.2, we have

$$|H_3(1)| \leq |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\,|a_4 - a_2a_3| + |a_5|\left|a_3 - a_2^2\right| \leq \frac{43}{576}.$$

If we compare the bound $\frac{43}{576}$ with the results of Zaprawa [48] and Deniz [13], see that this bound is better but not sharp.

3.1 Logarithmic Coefficients

The following formula defines the logarithmic coefficients β_n of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that belongs to $\mathcal{S}$

$$G_f(z) := \log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty}\beta_n z^n \quad \text{for } z \in \mathcal{U}. \tag{3.18}$$

In many estimations, these coefficients provide a significant contribution to the concept of univalent functions. In 1985, De Branges [10] proved that

$$\sum_{k=1}^{n} k\,(n-k+1)\,|\beta_n|^2 \leq \sum_{k=1}^{n} \frac{n-k+1}{k} \quad (n \geq 1) \tag{3.19}$$

and equality will be achieved if f has the form $z/\left(1-e^{i\theta}z\right)^2$ for some $\theta \in \mathbb{R}$. In its most comprehensive version, this inequality offers the famous Bieber-bach-Robertson-Milin conjectures regarding Taylor coefficients of $f \in \mathcal{S}$. We refer to [5, 16, 17] for further details on the proof of De Branges finding. By considering the logarithmic coefficients, Kayumov [22] was able to prove Brennan's conjecture for conformal mappings in 2005. For your reference, we mention a few works that have made major contributions to the research of the logarithmic coefficients. Andreev and Duren [4], Alimohammadi et al. [3], Deng [11], Roth [39], Ye [47], Obradović et al. [34], and finally the work of Girela [18] are the major contributions to the study of logarithmic coefficients for different subclasses of univalent functions.

As stated in the definition, it is simple to determine that for $f \in \mathcal{S}$, the logarithmic coefficients are computed by

$$\beta_1 = \frac{1}{2}a_2, \tag{3.20}$$

$$\beta_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{3.21}$$

$$\beta_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{3.22}$$

$$\beta_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right). \tag{3.23}$$

Theorem 3.4 *If $f \in \mathcal{S}_G^*$ and has the series representation $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ then*

$$|\beta_n| \leq \frac{1}{4n} \quad (n = 1, 2, 3, 4)\,. \tag{3.24}$$

These bounds are sharp and can be obtained from the extremal functions $f_i(z)$ $i = 1, 2, 3, 4$ given by (3.10)–(3.13).

Proof Let $f \in \mathcal{S}_G^*$. Then, putting (3.20), (3.21), (3.22) and (3.23) in (3.5), (3.6), (3.7) and (3.8), we get

$$\beta_1 = \frac{p_1}{8}, \tag{3.25}$$

$$\beta_2 = \frac{1}{192}\left(-7p_1^2 + 12p_2\right), \tag{3.26}$$

$$\beta_3 = \frac{1}{1152}\left(17p_1^3 - 56p_1p_2 + 48p_3\right), \tag{3.27}$$

$$\beta_4 = -\frac{1}{92160}\left(649p_1^4 + 1680p_2^2 + 3360p_1p_3 - 3060p_1^2p_2 - 2280p_4\right). \tag{3.28}$$

For β_1, using (2.3) in (3.25), we obtain

$$|\beta_1| \leq \frac{1}{4}.$$

For β_2, putting (2.5) in (3.26), we get

$$|\beta_2| \leq \frac{1}{8}.$$

For β_3, we can rewrite (3.27) as

$$|\beta_3| = \frac{1}{24}\left|p_3 - \frac{7}{6}p_1p_2 + \frac{17}{48}p_1^3\right|,$$

using (2.4) where $Q = \frac{7}{12}$ and $R = \frac{17}{48}$, we obtain

$$|\beta_3| \leq \frac{1}{12}.$$

For β_4, we can rewrite (3.28) as

$$\beta_4 = -\frac{1}{32}\left(\frac{649}{2880}p_1^4 + \frac{7}{12}p_2^2 + \frac{7}{6}p_1p_3 - \frac{17}{16}p_1^2p_2 - p_4\right). \tag{3.29}$$

Comparing the right side of (3.29) with inequality (2.7), where $\rho = \frac{649}{2880}$, $\varsigma = \frac{7}{12}$, $\tau = \frac{7}{12}$ and $\psi = \frac{17}{24}$. It follows that

$$8\varsigma\left(1-\varsigma\right)\left[\left(\tau\psi - 2\rho\right)^2 + \left(\tau\left(\varsigma+\tau\right) - \psi\right)^2\right] + \tau\left(1-\tau\right)\left(\psi - 2\varsigma\tau\right)^2 = 0.00442226$$

and

$$4\varsigma\tau^2\left(1-\tau\right)^2\left(1-\varsigma\right) = 0.057435.$$

Using (2.6) we deduce that

$$|\beta_4| \leq \frac{1}{16}.$$

□

3.2 Inverse Coefficients

It is well-known that the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$$

has an inverse f^{-1}, which is analytic in $|w| < 1/4$, as we know from Koebe's 1/4-theorem. If $f \in \mathcal{S}$, then

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots \quad (|w| < 1/4)\,. \tag{3.30}$$

Löwner [30] proved that, if $f \in \mathcal{S}$ and its inverse is given by (3.30), then the sharp estimate

$$|A_n| \leq \frac{(2n)!}{n!\,(n+1)!} \tag{3.31}$$

holds. It has been shown that the inverse of the Koebe function $k(z) = z/(1-z)^2$ provides the best bounds for all $|A_n| \quad (n = 2, 3, \ldots)$ in (3.31) over all members of $\mathcal{S}$. There has been a good deal of interest in determining the behavior of the inverse coefficients of f given in (3.30) when the corresponding function f is restricted to some proper geometric subclasses of $\mathcal{S}$. Alternate proofs of the inequality (3.31) have been given by several authors but a simpler proof was given by Yang [46].
Since $f\left(f^{-1}(w)\right) = w$, using (3.30) it's very clear to see

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= 2a_2^2 - a_3, \\ A_4 &= -5a_2^3 + 5a_2 a_3 - a_4. \end{aligned} \tag{3.32}$$

Theorem 3.5 *If $f \in \mathcal{S}_G^*$ and has the series representation $f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots$ then*

$$\begin{aligned} |A_2| &\leq \frac{1}{2}, \\ |A_3| &\leq \frac{5}{12}, \\ |A_4| &\leq \frac{31}{72}. \end{aligned} \tag{3.33}$$

These bounds are sharp, except A_4.

Proof Let $f \in \mathcal{S}_G^*$ and $f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots$. If equations (3.5), (3.6) and (3.7) are written in (3.32), we obtain

$$A_2 = -\frac{p_1}{4}, \tag{3.34}$$

$$A_3 = \frac{p_1^2}{6} - \frac{p_2}{8}, \tag{3.35}$$

$$A_4 = -\frac{1}{576}\left(48p_3 - 128p_1p_2 + 83p_1^3\right). \tag{3.36}$$

Because of $|p_1| \leq 2$, the result $|A_2| \leq \frac{1}{2}$ is trivial.
We now estimate $|A_3|$ by using (2.5) in equality (3.35). Thus, we have

$$|A_3| \leq \frac{1}{8}\left|p_2 - \frac{4}{3}p_1^2\right| \leq \frac{5}{12}.$$

From the relations Lemma 2.1 and (3.36), and by some simple calculations, we have

$$A_4 := K = -\frac{4-\sigma^2}{24}\left(\frac{31\sigma^3}{24\left(4-\sigma^2\right)} - \frac{5\sigma}{3}x - \frac{\sigma}{2}x^2 + \left(1-|x|^2\right)y\right),$$

where $\sigma = p_1 \in [0,2]$, $|x| \leq 1$ and $|y| \leq 1$. We now investigate upper bound of the $|K|$ according to σ.
A. If $\sigma = 0$, then $K = -\frac{1}{6}\left(1-|x|^2\right)y$ and so we have $|K| \leq \frac{1}{6}$.
B. Let $\sigma = 2$. Then $K = -\frac{31}{72}$ and so we have $|K| \leq \frac{31}{72}$.
C. We now assume that $\sigma \in (0,2)$. Then, we can write

$$\begin{aligned} |K| &= \frac{4-\sigma^2}{24}\left|\frac{31\sigma^3}{24\left(4-\sigma^2\right)} - \frac{5\sigma}{3}x - \frac{\sigma}{2}x^2 + \left(1-|x|^2\right)y\right| \\ &\leq \frac{4-\sigma^2}{24}\left(\left|\widetilde{A} + \widetilde{B}x + \widetilde{C}x^2\right| + 1 - |x|^2\right), \end{aligned} \tag{3.37}$$

where

$$\widetilde{A} = \frac{31\sigma^3}{24\left(4-\sigma^2\right)}, \quad \widetilde{B} = -\frac{5\sigma}{3} \text{ and } \widetilde{C} = -\frac{\sigma}{2}.$$

For the rest of the proof, we use Lemma 2.4. Then

$$\widetilde{A}\widetilde{C} = -\frac{31\sigma^4}{48\left(4-\sigma^2\right)} < 0.$$

C1. Note that the inequality

$$-4\widetilde{A}\widetilde{C}\left(\widetilde{C}^{-2} - 1\right) - \widetilde{B}^2 = \frac{31\sigma^4}{12\left(4-\sigma^2\right)}\left(\frac{4}{\sigma^2} - 1\right) - \frac{25\sigma^2}{9} \leq 0$$

which evidently holds for $s \in (-\infty, \infty)$. Moreover, the inequality $\left|\widetilde{B}\right| < 2\left(1 - \left|\widetilde{C}\right|\right)$ is equivalent to $\sigma < \frac{3}{4}$, which is true for $\sigma \in \left(0, \frac{3}{4}\right)$. Then, by (3.37) and Lemma 2.4,

we obtain

$$|K| \leq \frac{4-\sigma^2}{24}\left(1-\left|\widetilde{A}\right|+\frac{\widetilde{B}^2}{4\left(1-\left|\widetilde{C}\right|\right)}\right)$$
$$= \frac{7\sigma^3+128\sigma^2+288}{1728} \leq \frac{2581}{12288} \approx 0.210042 \tag{3.38}$$

for $\sigma \in \left(0, \frac{3}{4}\right)$.
C2. Since

$$4\left(1+\left|\widetilde{C}\right|\right)^2 = \sigma^2+4\sigma+4, \quad -4\widetilde{A}\widetilde{C}\left(\widetilde{C}^{-2}-1\right) = \frac{31\sigma^2}{12},$$

for $\sigma \in \left[\frac{3}{4}, 2\right)$, we see that the inequality

$$\frac{25\sigma^2}{9} = \widetilde{B}^2 < \min\left\{4\left(1+\left|\widetilde{C}\right|\right)^2, \ -4\widetilde{A}\widetilde{C}\left(\widetilde{C}^{-2}-1\right)\right\} = \frac{31\sigma^2}{12}$$

is equivalent to $\frac{7\sigma^2}{36} < 0$, which is false for $\sigma \in \left[\frac{3}{4}, 2\right)$.
C3. Observe that the inequality

$$\left|\widetilde{C}\right|\left(\left|\widetilde{B}\right|+4\left|\widetilde{A}\right|\right)-\left|\widetilde{A}\widetilde{B}\right| = \frac{\sigma^2\left(29\sigma^2-240\right)}{72\left(\sigma^2-4\right)} \leq 0$$

is false for $\sigma \in \left[\frac{3}{4}, 2\right)$.
C4. Note that the inequality

$$\left|\widetilde{A}\widetilde{B}\right|-\left|\widetilde{C}\right|\left(\left|\widetilde{B}\right|-4\left|\widetilde{A}\right|\right) = \frac{\sigma^2\left(401\sigma^2-240\right)}{72\left(4-\sigma^2\right)} \leq 0$$

is true $\sigma \in \left[\frac{3}{4}, 4\sqrt{\frac{15}{401}}\right]$. From (3.37) and Lemma 2.4, we get

$$|K| \leq \frac{4-\sigma^2}{24}\left(-\left|\widetilde{A}\right|+\left|\widetilde{B}\right|+\left|\widetilde{C}\right|\right)$$
$$= \frac{1}{576}\left(208\sigma-83\sigma^3\right) \leq \frac{3968}{1203}\sqrt{\frac{5}{1203}} \approx 0.212646 \tag{3.39}$$

for $\sigma \in \left[\frac{3}{4}, 4\sqrt{\frac{15}{401}}\right]$.

C5. It remains to consider the last case in Lemma 2.4, for $\sigma \in \left(4\sqrt{\frac{15}{401}}, 2\right)$. From (3.37), the inequality

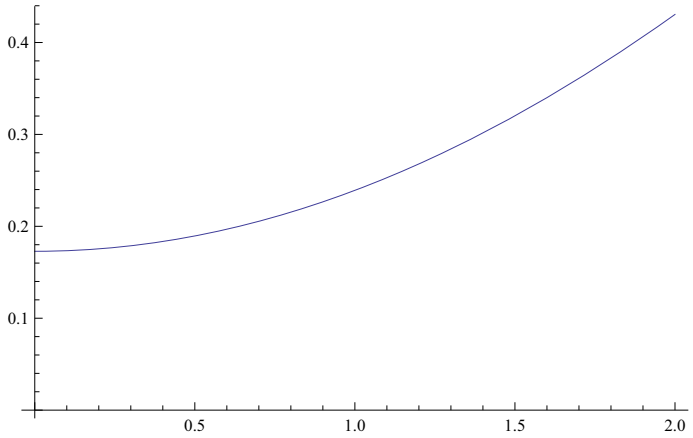


Fig. 2 The graph of the function $\Phi_1(\sigma)$ on $(0, 2)$

$$|K| \leq \frac{4-\sigma^2}{24}\left(\left|\widetilde{A}\right|+\left|\widetilde{C}\right|\right)\sqrt{1-\frac{\widetilde{B}^2}{4\widetilde{A}\widetilde{C}}}$$
$$= \frac{\sqrt{400-7\sigma^2}\left(19\sigma^2+48\right)}{576\sqrt{93}} \leq \frac{31}{72} \approx 0.430556 \tag{3.40}$$

is true for $\sigma \in \left(4\sqrt{\frac{15}{401}}, 2\right)$. In addition, as can be seen from the figure below, the function $\Phi_1(\sigma) = \frac{\sqrt{400-7\sigma^2}(19\sigma^2+48)}{576\sqrt{93}}$ is increasing in $\left(4\sqrt{\frac{15}{401}}, 2\right)$ and takes its maximum value at the limit (Fig. 2).

From A-C, we have $|A_3| \leq \frac{31}{72}$. □

4 Conclusions

In our present study, we have found the sharp upper bounds of the first six coefficients except $|a_6|$ and also of the Fekete–Szegö functional for functions in the class $\mathcal{S}_G^*$ of analytic functions which is related with starlike functions and generating function of Gregory coefficients which are new and not yet studied sofar in Geometric Functions Theory. Moreover, for the functions belonging to the considered class $\mathcal{S}_G^*$ were obtained the sharp bounds of logarithmic and inverse coefficients. Finally, the third Hankel determinant of defined this class was studied. We believe that new classes related to Gregory coefficients will be produced and will be the subject of much study in this field.

Author Contributions All authors actively worked on the research contained in the paper. All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

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