Computational and Applied Mathematics (2024) 43:335
https://doi.org/10.1007/s40314-024-02853-w

A new approach to soft relations and soft functions

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Received: 27 December 2023 / Revised: 1 April 2024 / Accepted: 7 July 2024 /
Published online: 18 July 2024
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Abstract
In this paper, a different approach is used to define a cartesian product on soft sets. This method processes both alternatives and parameters. The notion of the cartesian product is then used to define the idea of a soft relation. The concepts of reflexion, symmetry and transition are defined on the soft relation. Some properties are investigated. Also, the soft function notion is introduced. Various instances are provided as the key characteristics of the structures that are being presented are analyzed. Finally, an application is presented by building a decision making algorithm on the soft relation.

Keywords Soft cartesian product · Soft relations · Soft functions · Decision making

Mathematics Subject Classification 03E20 · 03E72 · 90B50

1 Introduction

The majority of real-world issues in social sciences, engineering, medical sciences, economics, etc. include facts that are of an erroneous type. These issues are resolved using mathematical concepts based on ambiguity and imprecision. Therefore, classical set theory, which is centered on the crisp and exact case, may not be entirely appropriate for tackling such uncertainty-related issues. Many hypotheses have been put out to cope with uncertainty in an effective manner. These include the theories of fuzzy sets (Zadeh 1965), intuitionistic fuzzy sets (Atanassov 1986), rough sets (Pawlak 1982), theory of interval mathematics (Atanassov 1994; Gorzalzany 1987). These ideas do, however, each have their own challenges. Molodtsov developed soft theory in 1999 (Molodtsov 1999). As a truly general mathematical tool for modeling uncertainties, Molodtsov suggested the soft set. Since there are no restrictions on how things may be described, researchers are free to pick the characteristics they require, which substantially facilitates decision-making and increases efficiency

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when only partial information is available. For modeling complex systems, a variety of mathematical techniques are available, including interval mathematics, fuzzy set theory, and probability theory. But each of these methods has its own unique set of challenges. Only systems that are stochastically stable allow for the use of probability theory. The interval mathematical approach is insufficiently flexible to deal with a variety of uncertainty. In fuzzy set theory, determining the value of the membership function has always been a challenge. Additionally, none of these strategies have parameterized tools, making it impossible for them to be successfully used to solving issues, particularly in the economic, environmental, and social problem domains. Soft set theory is different from other theories in that it doesn't have the aforementioned issues and has a larger range of multidimensional applications (John and Babitha 2014; Ozturk and Bayramov 2014; Ozturk and Yolcu 2016). Soft set theory offers a wide range of potential applications, some of which Molodtsov (Molodtsov 1999) reports in his work. In fields including the smoothness of functions, game theory, operation research, Riemann integration, and others, he effectively utilized soft set theory. Later, Maji et al. (2003) introduced new definitions of soft sets, such as a subset and a soft set's complement, and they thoroughly explained how to apply soft set theory to issues involving decision-making (Maji and Roy 2002). The approach of attribute reduction in rough set theory cannot be directly transferred to parameter reduction in soft set theory, as Chen et al. (2005) noted, although the specific steps involved in parameter reduction in soft set theory were not defined. Kostek (Kostek 1998) also made an effort to evaluate sound quality using a soft set methodology. Using the ideas of soft set theory, Mushrif et al. (2006) introduced a unique technique for classifying natural textures. The soft set theory is now moving forward quickly. Additionally, they deduced various features from the idea of soft groups and described it. Soft set theory was used by Feng et al. (2008) to address the algebraic structure of semi-rings. Jun (2008) also introduced and explored the idea of soft BCK/BCI-algebras. The use of soft sets in an ideal theory of BCK/BCI-algebras was studied by Jun and Park (2008). Various researches on soft cartesian product, soft relation concept etc. have been done before (Babitha and Sunil 2010, 2011; John 2021; Smarandache 2022; Taskopru and Karakose 2023; Yang and Guo 2011).

Despite the fact that relations are based on soft cartesian products, there isn't an explanation of the concept of relation that effectively employs both alternatives and parameters.Until recently, the soft structure's definitions of cartesian products, relations, and functions have all been essentially predicated on a single type of product. In other words, in previous soft cartesian product definitions, either only the parameters were multiplied or only the alternatives were multiplied. There are more available choices or parameters. This paper offers a solution to this problem. It is commonly known that the notion of function is based on the Cartesian product. This paper clearly defines the transition from soft cartesian product to soft relation and from soft relation to soft function. With the binary multiplication of both parameters and alternatives, this paper is rendered more functional in terms of soft relations and soft functions and will be a helpful tool for applications.

In this study, the concepts of cartesian product, relation and function, respectively, are defined by multiplying both alternatives and parameters. The rest of this paper is organized as follows. Some fundamental ideas, such as soft sets, are reviewed in Sect. 2. A novel method for soft cartesian products, soft relations, and soft functions is suggested in Sect. 3. Some significant theorems are also provided based on this new technique, and numerous examples are provided. In Sect. 4, we construct a decision making algorithm using the soft relation concept we have defined.

2 Preliminaries

Definition 1 (*Molodtsov* 1999) A pair (L, A) is called a soft set over a universe Z and with the set A of attributes from Σ, where $L : A \to P(Z)$ is a mapping. In other words, a soft set over Z is a parameterized family of subsets of the universe Z. For $\beta \in A$, $L(\beta)$ may be considered as the set of β-elements of the soft set (L, A), or as the set of β-approximate elements of the soft set. All soft sub-set over the (Z, Σ) denoted by $SS(Z, \Sigma)$.

Definition 2 (*Maji et al.* 2003) For two soft sets (L, A) and (S, B) over $SS(Z, \Sigma)$, we say that (L, A) is soft subset of (S, B), if

1. $A \subseteq B$
2. $\forall \beta \in A$, $F(\beta) \subseteq G(\beta)$.

We write $(L, A) \subseteq (S, B)$. (L, A) is said to be a soft super set of (S, B), if (S, B) is a soft subset of (L, A). We denote it by $(L, A) \supseteq (S, B)$.

Definition 3 (*Maji et al.* 2003) Union of two soft sets (L, A) and (S, B) over $SS(Z, \Sigma)$, is the soft set (N, C), where $C = A \cup B$, and $\forall \beta \in C$,

$$N(\beta) = \begin{cases} F(\beta) & if\ \beta \in A - B \\ G(\beta) & if\ \beta \in B - A \\ F(\beta) \cup G(\beta) & if\ \beta \in A \cap B \end{cases}$$

We write $(L, A)\widetilde{\cup}(S, B) = (N, C)$.

Definition 4 (*Irfan Ali et al.* 2009) Intersection of two soft sets (L, A) and (S, B) over $SS(Z, \Sigma)$, is the soft set (N, C), where $C = A \cap B$, and $\forall \beta \in C$,

$$N(\beta) = \begin{cases} F(\beta) & if\ \beta \in A - B \\ G(\beta) & if\ \beta \in B - A \\ F(\beta) \cup G(\beta) & if\ \beta \in A \cap B \end{cases}$$

We write $(L, A)\widetilde{\cap}(S, B) = (N, C)$.

Definition 5 (*Bayramov and Gunduz Aras* 2013) Let (L, Σ) be a soft set over $SS(Z, \Sigma)$, and $\sigma \in Z$.The soft set (L, A) is called a soft point, denoted by σ_β, if for the element $\beta \in \Sigma$, $L(\beta) = \{\sigma\}$ and $L(\beta') = \emptyset$ for all $\beta' \in \Sigma - \{\beta\}$.

3 Soft cartesian product and relations

Examining the previous definitions of soft cartesian products (Babitha and Sunil 2010; Peyghan et al. 2013), it is clear that either alternatives or parameters are simply multiplied. This causes some uncertainties, especially in practical application. Therefore, both parameters and alternatives are multiplied by the new definition below (see Definition 6)

Definition 6 Let $(L, A) \in SS(Z, \Sigma)$ and $(S, B) \in SS(Z', \Sigma')$ be two soft sets and $A \subseteq \Sigma$, $B \subseteq \Sigma'$. Then their cartesian product is $(L, A) \times (S, B) = (N, C)$ where $C = A \times B$, $N : A \times B \to SS(Z \times Z')$. Then (N, C) is defined as follows;

$$(N, C) = \left\{ \left((a, b), \langle (\sigma_i, \sigma_j) \rangle : \sigma_i \in L(a), \sigma_j \in S(b) \right) : a \in A, b \in B \right\}$$

Example 1 Let $Z = \{\sigma_1, \sigma_2, \sigma_3\}$, $Z' = \{\sigma_1', \sigma_2', \sigma_3'\}$ be a universe and $\Sigma = \{\beta_1, \beta_2, \beta_3\}$, $\Sigma' = \{\beta_1', \beta_2', \beta_3', \beta_4'\}$ and $A = \{\beta_1, \beta_2, \beta_3\} \subseteq \Sigma$, $B = \{\beta_1', \beta_2'\} \subseteq \Sigma'$. We define $(L, A) \in SS(Z, \Sigma)$ and $(S, B) \in SS(Z', \Sigma')$ as follows;

$$(L, A) = \{(\beta_1, (\sigma_1, \sigma_2, \sigma_3)), (\beta_2, (\sigma_1, \sigma_2, \sigma_3)), (\beta_3, (\sigma_1, \sigma_2, \sigma_3))\},$$
$$(S, B) = \{(\beta_1', (\sigma_1', \sigma_2', \sigma_3')), (\beta_2', (\sigma_1', \sigma_2'))\}.$$

Then $(L, A) \times (S, B) = (N, C)$ calculated as follows;

$$(N, C) = \left\{ \begin{array}{c} \left[(\beta_1, \beta_1'), \left(\begin{array}{c}(\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3')\end{array}\right)\right], \\ \left[(\beta_1, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right], \\ \left[(\beta_2, \beta_1'), \left(\begin{array}{c}(\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3')\end{array}\right)\right], \\ \left[(\beta_2, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right], \\ \left[(\beta_3, \beta_1'), \left(\begin{array}{c}(\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3')\end{array}\right)\right], \\ \left[(\beta_3, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right] \end{array} \right\}$$

Definition 7 Let $(L, A) \in SS(Z, \Sigma)$ and $(S, B) \in SS(Z', \Sigma')$ be two soft sets and $A \subseteq \Sigma$, $B \subseteq \Sigma'$. If (R, C) (briefly R) is a soft subset of $(L, A) \times (S, B)$, where $C \subseteq A \times B$ and $(R, C) \subseteq (L, A) \times (S, B)$, then (R, C) is called a soft relation on $(Z, \Sigma) \times (Z', \Sigma')$. All relations on the universe denoted by $SR(Z \times Z')_{\Sigma \times \Sigma'}$.

Example 2 We consider the Example 1. Then R is soft relation and defined as follows.

$$R = \left\{ \begin{array}{c} \left[(\beta_1, \beta_1'), \left(\begin{array}{c}(\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3')\end{array}\right)\right], \\ \left[(\beta_2, \beta_1'), \left(\begin{array}{c}(\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3')\end{array}\right)\right], \\ \left[(\beta_2, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right], \\ \left[(\beta_3, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right] \end{array} \right\}$$

Definition 8 Let $(L, A) \in SS(Z, \Sigma)$ and $(S, B) \in SS(Z', \Sigma')$ be two soft sets and $A \subseteq \Sigma$, $B \subseteq \Sigma'$ and let R be soft relation from (L, A) to (S, B). Then R^{-1} is a inverse soft relation from (S, B) to (L, A) and defined as follows.

$$R^{-1} = \left\{\left((b, a), \langle(\sigma_j, \sigma_i)\rangle : \sigma_i \in L(A), \sigma_j \in S(B)\right) : a \in A, b \in B\right\}$$

Example 3 We consider the Example 2 Then we get R^{-1} as follows.

$$R^{-1}=\left\{\begin{array}{c}\left[\left(\beta_1',\beta_1\right),\left(\begin{array}{c}\left(\sigma_1',\sigma_1\right),\left(\sigma_2',\sigma_1\right),\left(\sigma_3',\sigma_1\right),\left(\sigma_1',\sigma_2\right),\left(\sigma_2',\sigma_2\right),\\ \left(\sigma_3',\sigma_2\right),\left(\sigma_1',\sigma_3\right),\left(\sigma_2',\sigma_3\right),\left(\sigma_3',\sigma_3\right)\end{array}\right)\right],\\ \left[\left(\beta_1',\beta_2\right),\left(\begin{array}{c}\left(\sigma_1',\sigma_1\right),\left(\sigma_2',\sigma_1\right),\left(\sigma_3',\sigma_1\right),\left(\sigma_1',\sigma_2\right),\left(\sigma_2',\sigma_2\right),\\ \left(\sigma_3',\sigma_2\right),\left(\sigma_1',\sigma_3\right),\left(\sigma_2',\sigma_3\right),\left(\sigma_3',\sigma_3\right)\end{array}\right)\right],\\ \left[\left(\beta_2',\beta_2\right),\left(\left(\sigma_1',\sigma_1\right),\left(\sigma_2',\sigma_1\right),\left(\sigma_1',\sigma_2\right),\left(\sigma_2',\sigma_2\right),\left(\sigma_1',\sigma_3\right),\left(\sigma_2',\sigma_3\right)\right)\right],\\ \left[\left(\beta_2',\beta_2\right),\left(\left(\sigma_1',\sigma_1\right),\left(\sigma_2',\sigma_1\right),\left(\sigma_1',\sigma_2\right),\left(\sigma_2',\sigma_2\right),\left(\sigma_1',\sigma_3\right),\left(\sigma_2',\sigma_3\right)\right)\right]\end{array}\right\}$$

Proposition 1 *Let (L, A) and (S, B) be two soft sets over the common universe $SS(Z, \Sigma)$ and R be soft relation from (L, A) to (S, B).Then R^{-1} is a soft relation from (S, B) to (L, A).*

Proof For $\sigma_i \in L(A)$, $\sigma_j \in S(B)$, $a \in A$ and $b \in B$, we have

$$R^{-1}=\left\{\left((b,a),\langle\left(\sigma_j,\sigma_i\right)\rangle:\sigma_j\in S(B),\ \sigma_i\in L(A),\right):a\in A,b\in B\right\}$$

and $R^{-1} \subseteq (S, B) \times (L, A)$. Hence, $R(a, b) = R^{-1}(b, a)$. Then, R^{-1} is a soft relation from (S, B) to (L, A). □

Proposition 2 *Let (L, A) and (S, B) be two soft sets over the common universe $SS(Z, \Sigma)$ and R_1, R_2 be soft relation from (L, A) to (S, B). Then*

1. $\left(R_1^{-1}\right)^{-1}=R_1$
2. $R_1 \subseteq R_2 \Rightarrow R_1^{-1} \subseteq R_2^{-1}$

Proof 1. Let $\sigma_i \in L(A)$, $\sigma_j \in S(B)$, $a \in A$ and $b \in B$. From the Proposition 1, $R_1(a, b) = R_1^{-1}(b, a)$. Then we have, $R_1(a, b) = R_1^{-1}(b, a) = \left(R_1^{-1}\right)^{-1}(b, a)$. It follows that $\left(R_1^{-1}\right)^{-1} = R_1$.

2. If $R_1 \subseteq R_2$, then $R_1(a, b) \subseteq R_2(a, b)$. Thus $R_1(a, b) = R_1^{-1}(b, a) \subseteq R_2(a, b) = R_2^{-1}(b, a)$. So, $R_1^{-1}(b, a) \subseteq R_2^{-1}(b, a)$. Hence, we have $R_1^{-1} \subseteq R_2^{-1}$. This completes of the condition. □

Definition 9 Let R and S two soft relation on $(L, A) \in SS(Z, \Sigma)$ and $a, b \in A$. Then;

1. The union of two soft relation R and S, denoted as $R \cup S$ is defined by

$$R \cup S = \{L(a) \times L(b) \,|L(a) \times L(b) \in R \text{ or } L(a) \times L(b) \in S\}$$

2. The union of two soft relation R and S, denoted as $R \cap S$ is defined by

$$R \cap S = \{L(a) \times L(b) \,|L(a) \times L(b) \in R \text{ and } L(a) \times L(b) \in S\}$$

Example 4 We consider the Example 1. Then R and S are soft relation and defined as follows.

$$R=\left\{\begin{array}{c}\left[\left(\beta_1,\beta_1'\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_1,\sigma_3'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\\ \left(\sigma_2,\sigma_3'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right),\left(\sigma_3,\sigma_3'\right)\end{array}\right)\right],\\ \left[\left(\beta_2,\beta_1'\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_1,\sigma_3'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\\ \left(\sigma_2,\sigma_3'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right),\left(\sigma_3,\sigma_3'\right)\end{array}\right)\right],\\ \left[\left(\beta_2,\beta_2'\right),\left(\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right)\right)\right],\\ \left[\left(\beta_3,\beta_2'\right),\left(\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right)\right)\right]\end{array}\right\}$$

$$S = \left\{ \begin{array}{c} \left[(\beta_1, \beta_1'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'))\right], \\ \left[(\beta_1, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right], \\ \left[(\beta_2, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right], \\ \left[(\beta_3, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right] \end{array} \right\}$$

Then;

$$R \cup S = \left\{ \begin{array}{c} \left[(\beta_1, \beta_1'), \left(\begin{array}{c} (\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3') \end{array} \right)\right], \\ \left[(\beta_1, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right], \\ \left[(\beta_2, \beta_1'), \left(\begin{array}{c} (\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3') \end{array} \right)\right], \\ \left[(\beta_2, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right], \\ \left[(\beta_3, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right] \end{array} \right\}$$

$$R \cap S = \left\{ \begin{array}{c} \left[(\beta_1, \beta_1'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'))\right], \\ \left[(\beta_2, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right] \\ \left[(\beta_3, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right] \end{array} \right\}$$

Definition 10 Let Z, Z', Z'' be a universes and Σ, Σ', Σ' be a parameter sets. (L, A), (S, B) and (N, E) be soft sets over the common universe Z, Z', Z'' respectively. and $A \in \Sigma,\ B \in \Sigma', E \in \Sigma''$. Let R and S be two soft relations from (L, A) to (S, B) and (S, B) to (N, E), respectively. Then their soft composition (briefly composition) is defined as the soft set $(T, K) = (R, C) \circ (S, D)$, where $K = C \circ D = \{\{a, e\} \in A \times E; \exists b \in B, ((a, b) \in C \wedge (b, e) \in D)\}$ and $T(a, e) = R(a) \diamond S(e)$ for all $(a, e) \in K$. Note that $R(\pi) \diamond S(\mu)$ denotes the ordinary composition on the universe. Then, we have

$$R \circ S = \left\{ \left((a, e), \langle \left(\sigma_i, \sigma_j\right) \rangle : \sigma_i \in L(a), \sigma_j \in N(e) \right) : (a, e) \in K \right\}$$

Example 5 We consider the Example 1.Let $Z'' = \left\{\sigma_1'', \sigma_2'', \sigma_3''\right\}$ be a universe and $\Sigma'' = \left\{\beta_1'', \beta_2'', \beta_3''\right\}$ be a parameter set and $(D, C) \in SS(Z'', \Sigma'')$. (D, C) be given as follows.

$$(D, C) = \left\{ (\beta_1'', \left(\sigma_1'', \sigma_2'', \sigma_3''\right), (\beta_2'', \left(\sigma_1'', \sigma_2'', \sigma_3''\right) \right\}$$

Let $R \subseteq (L, A) \times (S, B)$, $S \subseteq (S, B) \times (D, C)$ relations be defined as below.

$$R = \left\{ \begin{array}{c} \left[(\beta_1, \beta_1'), \left(\begin{array}{c} (\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3') \end{array} \right)\right], \\ \left[(\beta_2, \beta_1'), \left(\begin{array}{c} (\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_1, \sigma_3'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), \\ (\sigma_2, \sigma_3'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'), (\sigma_3, \sigma_3') \end{array} \right)\right], \\ \left[(\beta_2, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right], \\ \left[(\beta_3, \beta_2'), ((\sigma_1, \sigma_1'), (\sigma_1, \sigma_2'), (\sigma_2, \sigma_1'), (\sigma_2, \sigma_2'), (\sigma_3, \sigma_1'), (\sigma_3, \sigma_2'))\right] \end{array} \right\}$$

$$S=\left\{\begin{array}{l}\left[\left(\beta_1',\beta_1''\right),\left(\begin{array}{c}\left(\sigma_1',\sigma_1''\right),\left(\sigma_1',\sigma_2''\right),\left(\sigma_1',\sigma_3''\right),\left(\sigma_2',\sigma_1''\right),\left(\sigma_2',\sigma_2''\right),\\ \left(\sigma_2',\sigma_3''\right),\left(\sigma_3',\sigma_1''\right),\left(\sigma_3',\sigma_2''\right),\left(\sigma_3',\sigma_3''\right)\end{array}\right)\right],\\ \left[\left(\beta_2',\beta_1''\right),\left(\begin{array}{c}\left(\sigma_1',\sigma_1''\right),\left(\sigma_1',\sigma_2''\right),\left(\sigma_1',\sigma_3''\right),\left(\sigma_2',\sigma_1''\right),\left(\sigma_2',\sigma_2''\right),\\ \left(\sigma_2',\sigma_3''\right)\end{array}\right)\right],\\ \left[\left(\beta_1',\beta_2''\right),\left(\begin{array}{c}\left(\sigma_1',\sigma_1''\right),\left(\sigma_1',\sigma_2''\right),\left(\sigma_1',\sigma_3''\right),\left(\sigma_2',\sigma_1''\right),\left(\sigma_2',\sigma_2''\right),\\ \left(\sigma_2',\sigma_3''\right),\left(\sigma_3',\sigma_1''\right),\left(\sigma_3',\sigma_2''\right),\left(\sigma_3',\sigma_3''\right)\end{array}\right)\right],\\ \left[\left(\beta_2',\beta_2''\right),\left(\begin{array}{c}\left(\sigma_1',\sigma_1''\right),\left(\sigma_1',\sigma_2''\right),\left(\sigma_1',\sigma_3''\right),\left(\sigma_2',\sigma_1''\right),\left(\sigma_2',\sigma_2''\right),\\ \left(\sigma_2',\sigma_3''\right)\end{array}\right)\right]\end{array}\right\}.$$

The parameters of $R\circ S$ system is provided as;

$$\begin{aligned}\left\{\left(\beta_1,\beta_1'\right)\circ\left(\beta_1',\beta_1''\right)\right\}&\Rightarrow\left(\beta_1,\beta_1''\right),\\ \left\{\left[\left(\beta_2,\beta_2'\right)\circ\left(\beta_2',\beta_1''\right)\right],\left[\left(\beta_2,\beta_1'\right)\circ\left(\beta_1',\beta_1''\right)\right]\right\}&\Rightarrow\left(\beta_2,\beta_1''\right),\\ \left\{\left(\beta_1,\beta_1'\right)\circ\left(\beta_1',\beta_2''\right)\right\}&\Rightarrow\left(\beta_1,\beta_2''\right)\\ \left\{\left[\left(\beta_2,\beta_1'\right)\circ\left(\beta_1',\beta_2''\right)\right],\left[\left(\beta_2,\beta_2'\right)\circ\left(\beta_2',\beta_2''\right)\right]\right\}&\Rightarrow\left(\beta_2,\beta_2''\right),\\ \left\{\left(\beta_3,\beta_2'\right)\circ\left(\beta_2',\beta_1''\right)\right\}&\Rightarrow\left(\beta_3,\beta_1''\right),\\ \left\{\left(\beta_3,\beta_2'\right)\circ\left(\beta_2',\beta_2''\right)\right\}&\Rightarrow\left(\beta_3,\beta_2''\right)\end{aligned}$$

Then, we calculated $R\circ S$ as follows;

$$R\circ S=\left\{\begin{array}{l}\left[\left(\beta_1,\beta_1''\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1''\right),\left(\sigma_1,\sigma_2''\right),\left(\sigma_1,\sigma_3''\right),\left(\sigma_2,\sigma_1''\right),\left(\sigma_2,\sigma_2''\right),\\ \left(\sigma_2,\sigma_3''\right),\left(\sigma_3,\sigma_1''\right),\left(\sigma_3,\sigma_2''\right),\left(\sigma_3,\sigma_3''\right)\end{array}\right)\right],\\ \left[\left(\beta_2,\beta_1''\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1''\right),\left(\sigma_1,\sigma_2''\right),\left(\sigma_1,\sigma_3''\right),\left(\sigma_2,\sigma_1''\right),\left(\sigma_2,\sigma_2''\right),\\ \left(\sigma_2,\sigma_3''\right),\left(\sigma_3,\sigma_1''\right),\left(\sigma_3,\sigma_2''\right),\left(\sigma_3,\sigma_3''\right)\end{array}\right)\right],\\ \left[\left(\beta_1,\beta_2''\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1''\right),\left(\sigma_1,\sigma_2''\right),\left(\sigma_1,\sigma_3''\right),\left(\sigma_2,\sigma_1''\right),\left(\sigma_2,\sigma_2''\right),\\ \left(\sigma_2,\sigma_3''\right),\left(\sigma_3,\sigma_1''\right),\left(\sigma_3,\sigma_2''\right),\left(\sigma_3,\sigma_3''\right)\end{array}\right)\right],\\ \left[\left(\beta_2,\beta_2''\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1''\right),\left(\sigma_1,\sigma_2''\right),\left(\sigma_1,\sigma_3''\right),\left(\sigma_2,\sigma_1''\right),\left(\sigma_2,\sigma_2''\right),\\ \left(\sigma_2,\sigma_3''\right),\left(\sigma_3,\sigma_1''\right),\left(\sigma_3,\sigma_2''\right),\left(\sigma_3,\sigma_3''\right)\end{array}\right)\right],\\ \left[\left(\beta_3,\beta_1''\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1''\right),\left(\sigma_1,\sigma_2''\right),\left(\sigma_1,\sigma_3''\right),\left(\sigma_2,\sigma_1''\right),\left(\sigma_2,\sigma_2''\right),\\ \left(\sigma_2,\sigma_3''\right),\left(\sigma_3,\sigma_1''\right),\left(\sigma_3,\sigma_2''\right),\left(\sigma_3,\sigma_3''\right)\end{array}\right)\right],\\ \left[\left(\beta_3,\beta_2''\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1''\right),\left(\sigma_1,\sigma_2''\right),\left(\sigma_1,\sigma_3''\right),\left(\sigma_2,\sigma_1''\right),\left(\sigma_2,\sigma_2''\right),\\ \left(\sigma_2,\sigma_3''\right),\left(\sigma_3,\sigma_1''\right),\left(\sigma_3,\sigma_2''\right),\left(\sigma_3,\sigma_3''\right)\end{array}\right)\right]\end{array}\right\}$$

Definition 11 Let Z be a universe and Σ be a parameter sets.The relation R on $(L,A)\in SS(Z,\Sigma)$ is called soft reflexive if $R(\beta)$ is a classic reflexive relation for all $\beta\in A$ with $R(\beta)\neq\varnothing$.

Definition 12 Let Z be a universe and Σ be a parameter sets.The relation R on $(L,A)\in SS(Z,\Sigma)$ is called soft symmetric if $R=R^{-1}$.

Definition 13 Let Z be a universe and Σ be a parameter sets.The relation R on $(L, A) \in SS(Z, \Sigma)$ is called soft transitive if $R \circ R \subseteq R$.

Definition 14 Let Z be a universe and Σ be a parameter sets.The relation R on $(L, A) \in SS(Z, \Sigma)$ is called to be soft equivalance relation if the following conditions are hold.

i) R is reflexive if $\sigma_\beta R \sigma_\beta$, $\forall \sigma_\beta \in (L, A)$,
ii) R is symmetric if $\sigma_{1\beta_1} R \sigma_{2\beta_2} \Rightarrow \sigma_{2\beta_2} R \sigma_{1\beta_1}$, $\forall \sigma_{1\beta_1}, \sigma_{2\beta_2} \in (L, A)$,
iii) R is transitive if $\sigma_{1\beta_1} R \sigma_{2\beta_2}$ and $\sigma_{2\beta_2} R \sigma_{3\beta_3} \Rightarrow \sigma_{1\beta_1} R \sigma_{3\beta_3}$, $\forall \sigma_{1\beta_1}, \sigma_{2\beta_2}, \sigma_{3\beta_3} \in (L, A)$.

Definition 15 Let R be an equivalence relation on a soft set $(L, A) \in SS(Z, \Sigma)$ and $\beta \in A$. Then soft equivalence class of $\sigma_\beta \in (L, A)$, denoted by $[\sigma_\beta]$, is defined as $[\sigma_\beta] = \{\sigma'_{\beta'} : \sigma_\beta R \sigma'_{\beta'}\}$. The set of $\{[\sigma_\beta] : \sigma_\beta \in (L, A)\}$ is called the quotient soft set of (L, A) and denoted by $(L, A)/R$.

Lemma 1 *Let R be an equivalence relation on a soft set $(L, A) \in SS(Z, \Sigma)$ and $\beta_1, \beta_2 \in A$. For any $\sigma_{1\beta_1}, \sigma_{2\beta_2} \in (L, A)$, $\sigma_{1\beta_1} RL(\beta_2)$ iff $[f(a)] = [f(b)]$.*

Proof Suppose $[\sigma_{1\beta_1}] = [\sigma_{2\beta_2}]$. Since R is reflexive $\sigma_{2\beta_2} R \sigma_{2\beta_2}$. Hence $\sigma_{2\beta_2} \in [\sigma_{2\beta_2}] = [\sigma_{2\beta_2}]$ which gives $\sigma_{1\beta_1} R \sigma_{2\beta_2}$.

Conversely, suppose $\sigma_{1\beta_1} R \sigma_{2\beta_2}$. Let $\sigma_{1\beta_1} \in [\sigma_{1\beta_1}]$. Then $\sigma_{1\beta_1} \in \sigma_{1\beta_1}$. Using the transitive property of R this gives $\sigma_{1\beta_1} \in [\sigma_{2\beta_2}]$. Hence $[\sigma_{1\beta_1}] \subseteq [\sigma_{2\beta_2}]$. Using a similar argument $[\sigma_{2\beta_2}] \subseteq [\sigma_{1\beta_1}]$. Hence $[\sigma_{1\beta_1}] \subseteq [\sigma_{2\beta_2}]$. □

Definition 16 A collection $P = \{(L_i, A_i) : i \in I\}$ of nonempty soft subset of soft set $(L, A) \in SS(Z, \Sigma)$ is called a soft partition of (L, A) if

i) $\tilde{\cup}_{i \in I} (L_i, A_i) = (L, A)$,
ii) $A_i \cap A_j = \emptyset$ whenever $i \neq j$. The members of partition are called a block of (L, A).

Theorem 3 *Corresponding to every equivalence relation defined on a soft set (L, A). There exists a partition on (L, A) and this partition precisely consists of the equivalence classes of R.*

Proof Let $[\sigma_\beta]$ be a equivalence class w.r.t a relation R on (L, A). Let A_γ denote all those elements in A corresponding to $[\sigma_\beta]$. $A_\gamma = \{\beta_2 \in A : \sigma_{2\beta_2} R \sigma_{1\beta_1}\}$. Thus we can denote $[\sigma_\beta]$ as (L, A_γ). So we have to show that the collection $\{(L, A_\gamma) : \beta_1 \in A\}$ of such distinct sets forms a partition P of (L, A). In order to prove this we should prove

1. $\tilde{\cup}_{i \in I} (L_i, A_i) = (L, A)$
2. $A_i \cap A_j = \emptyset$ whenever $i \neq j$

Since R is reflexive $\sigma_\beta R \sigma_\beta$ $\forall \beta \in A$ so that part (1) can prove easily. Now for the second part, Let $x \in A_i \cap A_j$ then $\sigma_\alpha \in (L, A_i)$ and $\sigma_\alpha \in (L, A_j)$ then $\sigma_\alpha R \sigma_{\beta_i}$ and $\sigma_\alpha R \sigma_{\beta_j}$. Using transitive property of R we have $L(A_i) RL(A_j)$. Now using the Lemma 1, we have $[\sigma_{1\beta_1}] = [\sigma_{2\beta_2}]$. This gives $\sigma_{1\beta_1} = \sigma_{2\beta_2}$. □

Example 6 Consider a soft set $(L, A) \in SS(Z, \Sigma)$ where $Z = \{1, 2, 3, ..., 31\}$ be month days, $\Sigma = \{\beta_1 = January, \beta_2 = February, \beta_3 = March, ..., \beta_{12} = December\}$ be a months.

$(L, A) = \{(\beta_1, (1, 2, ..., 31)), (\beta_2, (1, 2, ..., 28)), (\beta_3, (1, 2, ..., 31)), ..., (\beta_{12}, (1, 2, ..., 31))\}$

Then;

$$\begin{aligned} L(\beta_1) &= \{1, 2, ..., 31\}, \\ L(\beta_2) &= \{1, 2, ..., 28\}, \\ L(\beta_3) &= \{1, 2, ..., 31\}, \\ L(\beta_4) &= \{1, 2, ..., 30\}, \\ &\vdots \\ L(\beta_{12}) &= \{1, 2, ..., 31\}, \end{aligned}$$

where $L(\beta_1), L(\beta_2), ..., L(\beta_{12})$ represents the days of January, February,...,December, respectively. If we examine it pointwise, 1_{β_1}, means the $1st$ of January. Let $R \subseteq (L, A) \times (L, A)$ relation be defined as days of 2018 year that coincide with the same days. Then, the relation R is clearly satisfied reflexive,symmetric and transitive properties. That is R is a equivalence relation. Let $(L, A)^{Mon}$, $(L, A)^{Tue}$, ... represents the days of the year corresponding to Monday,Tuesday,..., respectively, in the 2018 year. Now, we find equivalence classes.

$$\begin{aligned} [(L, A)^{Mon}] &= \{1_{\beta_1}, ..., 29_{\beta_1}, 5_{\beta_2}, ..., 26_{\beta_2}, ..., 3_{\beta_{12}}, ..., 31_{\beta_{12}}\}, \\ [(L, A)^{Tue}] &= \{2_{\beta_1}, ..., 30_{\beta_1}, 6_{\beta_2}, ..., 27_{\beta_2}, ..., 4_{\beta_{12}}, ..., 25_{\beta_{12}}\}, \\ [(L, A)^{Wed}] &= \{3_{\beta_1}, ..., 31_{\beta_1}, 7_{\beta_2}, ..., 28_{\beta_2}, ..., 5_{\beta_{12}}, ..., 26_{\beta_{12}}\}, \\ [(L, A)^{Thu}] &= \{4_{\beta_1}, , ..., 25_{\beta_1}, 8_{\beta_2}, ..., 22_{\beta_2}, ..., 6_{\beta_{12}}, ..., 27_{\beta_{12}}\}, \\ &\vdots \\ [(L, A)^{Sun}] &= \{7_{\beta_1}, ..., 28_{\beta_1}, 4_{\beta_2}, ..., 25_{\beta_2}, ..., 2_{\beta_{12}}, ..., 30_{\beta_{12}}\} \end{aligned}$$

Then, we obtain

$$(L, A)/R = \{(L, A)^{Mon}, (L, A)^{Tue}, ..., (L, A)^{Sun}\}$$

It is clear that

$$(L, A) = [(L, A)^{Mon}] \cup [(L, A)^{Tue}] \cup ... \cup [(L, A)^{Sun}].$$

Definition 17 Let I be a relation on $(L, A) \in SS(Z, \Sigma)$. If $I = \left\{(\sigma_\beta, \sigma_\beta) : \sigma_\beta \in SS(Z, \Sigma)\right\}$, then I is called the identity soft relation. It is clear that if R is a soft reflexive relation on (L, A) if and only if $I \subset R$.

Definition 18 Let R be a relation on $(L, A) \in SS(Z, \Sigma)$. Reflexive closure of R, indicated by $\overline{clr}(R)$, is the smallest reflexive relation containing R.

Definition 19 Let R be a relation on $(L, A) \in SS(Z, \Sigma)$. Symmetric closure of R, indicated by $\overline{cls}(R)$, is the smallest symmetric relation containing R.

Definition 20 Let R be a relation on $(L, A) \in SS(Z, \Sigma)$. The transitive closure of R, indicated by $\overline{clt}(R) = \bigcup_{n=1}^{\infty} R_n$, where R^n denotes the nth power of soft relation of R on Z such that $R_1 = R$ and $R_n = R_{n-1} \circ R$ with

$$R \circ R = \left\{\left(\sigma_{1\beta_1}, \sigma_{2\beta_2}\right) : \exists \sigma_{3\beta_3} \in (L, A) \ni \left(\sigma_{1\beta_1}, \sigma_{2\beta_2}\right), \left(\sigma_{2\beta_2}, \sigma_{3\beta_3}\right) \in R\right\}.$$

Proposition 4 *Let R be a relation on $(L, A) \in SS(Z, \Sigma)$. Then*

1. $\overline{clr}(R) = R \cup I$.
2. $\overline{cls}(R) = R \cup R^{-1}$

Proof (1) $R \cup I \supset R$. $(\sigma_\beta, \sigma_\beta) \in I \subset R \cup I$. so $R \cup I$ is reflexive. On the other hand, if S is a reflexive soft relation on (L, A) and $R \subset S$. By the reflexivity of S, $I \subset S$, Thus we have $R \cup I \subset S$. So $\overline{clr}(R) = R \cup I$.

(2) $\left(R \cup R^{-1}\right)^{-1} = R^{-1} \cup \left(R^{-1}\right)^{-1} = R^{-1} \cup R = R \cup R^{-1}$. $R \cup R^{-1}$ is a symmetric soft relation on (L, A) and $R \subset R \cup R^{-1}$. If S is a symmetric soft relation on (L, A) and $R \subset S$. Then $R^{-1} \subset S^{-1}$ and $S = S^{-1} \supset R \cup R^{-1}$. Thus, $\overline{cls}(R) = R \cup R^{-1}$.

□

Example 7 Let $Z = \{\sigma_1, \sigma_2\}$, be a universe and $\Sigma = \{\beta_1, \beta_2\}$, $A = \{\beta_1, \beta_2, \beta_3\} \subseteq \Sigma$. We define $(L, A) \in SS(Z, \Sigma)$ as follows;

$$(L, A) = \{(\beta_1, (\sigma_1, \sigma_2)), (\beta_2, (\sigma_1, \sigma_2))\},$$

if we write the set in soft point form,

$$(L, A) = \left\{\sigma_{1\beta_1}, \sigma_{1\beta_2}, \sigma_{2\beta_1}, \sigma_{2\beta_2}\right\},$$

Consider a soft relation R on (L, A) as

$$R = \left\{ \begin{array}{l} [(\beta_1, \beta_1), ((\sigma_1, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (\sigma_2, \sigma_2))], \\ [(\beta_1, \beta_2), ((\sigma_1, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (\sigma_2, \sigma_2))], \\ [(\beta_2, \beta_1), ((\sigma_1, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (\sigma_2, \sigma_2))], \\ [(\beta_2, \beta_2), ((\sigma_1, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (\sigma_2, \sigma_2))], \end{array} \right\}$$

if we write the set in soft point form,

$$R = \left\{ \begin{array}{llll} \left(\sigma_{1\beta_1}, \sigma_{1\beta_1}\right), & \left(\sigma_{1\beta_1}, \sigma_{1\beta_2}\right), & \left(\sigma_{1\beta_1}, \sigma_{2\beta_1}\right), & \left(\sigma_{1\beta_1}, \sigma_{2\beta_2}\right), \\ \left(\sigma_{1\beta_2}, \sigma_{1\beta_1}\right), & \left(\sigma_{1\beta_2}, \sigma_{1\beta_2}\right), & \left(\sigma_{1\beta_2}, \sigma_{2\beta_1}\right), & \left(\sigma_{1\beta_2}, \sigma_{2\beta_2}\right), \\ \left(\sigma_{2\beta_1}, \sigma_{1\beta_1}\right), & \left(\sigma_{2\beta_1}, \sigma_{1\beta_2}\right), & \left(\sigma_{2\beta_1}, \sigma_{2\beta_1}\right), & \left(\sigma_{2\beta_1}, \sigma_{2\beta_2}\right), \\ \left(\sigma_{2\beta_2}, \sigma_{1\beta_1}\right), & \left(\sigma_{2\beta_2}, \sigma_{1\beta_2}\right), & \left(\sigma_{2\beta_2}, \sigma_{2\beta_1}\right), & \left(\sigma_{2\beta_2}, \sigma_{2\beta_2}\right) \end{array} \right\}$$

Also, It is clear that I is written as follows.

$$I = \left\{\left(\sigma_{1\beta_1}, \sigma_{1\beta_1}\right), \left(\sigma_{1\beta_2}, \sigma_{1\beta_2}\right), \left(\sigma_{2\beta_1}, \sigma_{2\beta_1}\right), \left(\sigma_{2\beta_2}, \sigma_{2\beta_2}\right)\right\},$$

$$I = \left\{ \begin{array}{l} [(\beta_1, \beta_1), ((\sigma_1, \sigma_1), (\sigma_2, \sigma_2))], \\ [(\beta_2, \beta_2), ((\sigma_1, \sigma_1), (\sigma_2, \sigma_2))], \end{array} \right\} \text{(Soft Set Form)}$$

Then

$$\overline{clr}(R) = R \cup I$$

$$= \left\{ \begin{array}{llll} \left(\sigma_{1\beta_1}, \sigma_{1\beta_1}\right), & \left(\sigma_{1\beta_1}, \sigma_{1\beta_2}\right), & \left(\sigma_{1\beta_1}, \sigma_{2\beta_1}\right), & \left(\sigma_{1\beta_1}, \sigma_{2\beta_2}\right), \\ \left(\sigma_{1\beta_2}, \sigma_{1\beta_1}\right), & \left(\sigma_{1\beta_2}, \sigma_{1\beta_2}\right), & \left(\sigma_{1\beta_2}, \sigma_{2\beta_1}\right), & \left(\sigma_{1\beta_2}, \sigma_{2\beta_2}\right), \\ \left(\sigma_{2\beta_1}, \sigma_{1\beta_1}\right), & \left(\sigma_{2\beta_1}, \sigma_{1\beta_2}\right), & \left(\sigma_{2\beta_1}, \sigma_{2\beta_1}\right), & \left(\sigma_{2\beta_1}, \sigma_{2\beta_2}\right), \\ \left(\sigma_{2\beta_2}, \sigma_{1\beta_1}\right), & \left(\sigma_{2\beta_2}, \sigma_{1\beta_2}\right), & \left(\sigma_{2\beta_2}, \sigma_{2\beta_1}\right), & \left(\sigma_{2\beta_2}, \sigma_{2\beta_2}\right) \end{array} \right\}$$

$$\overline{cls}(R) = R \cup R^{-1}$$

$$= \left\{ \begin{array}{l} [(\beta_1, \beta_1), ((\sigma_1, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (\sigma_2, \sigma_2))], \\ [(\beta_1, \beta_2), ((\sigma_1, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (\sigma_2, \sigma_2))], \\ [(\beta_2, \beta_1), ((\sigma_1, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (\sigma_2, \sigma_2))], \\ [(\beta_2, \beta_2), ((\sigma_1, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (\sigma_2, \sigma_2))], \end{array} \right\}$$

It is clear that R is both reflexive and symetric relation. The sets given in the example are also shown as points to make it easier for the reader to understand the concepts.

Proposition 5 *Let (L, A) over the universe $SS(Z, \Sigma)$ and R, S be soft relation on (L, A). The reflexive closure operator $\overline{clr}(R)$ has the following properties:*

1. $\forall R \in SS(L, A)$, $R \subset \overline{clr}(R)$,
2. $\forall R, S \in SS(L, A)$, $\overline{clr}(R \cup S) = \overline{clr}(R) \cup \overline{clr}(S)$, $\overline{clr}(R \cap S) = \overline{clr}(R) \cap \overline{clr}(S)$,
3. $\forall R, S \in SS(L, A)$, *If* $R \subset S$, *then* $\overline{clr}(R) \subset \overline{clr}(S)$,
4. $\forall R \in SS(L, A)$, $\overline{clr}\left(\overline{clr}(R)\right) = \overline{clr}(R)$.

Proposition 6 *Let (L, A) over the universe $SS(Z, \Sigma)$ and R, S be soft relation on (L, A). The symmetric closure operator $\overline{cls}(R)$ has the following properties:*

1. $\forall R \in SS(L, A)$, $R \subset \overline{cls}(R)$,
2. $\forall R, S \in SS(L, A)$, $\overline{cls}(R \cup S) = \overline{cls}(R) \cup \overline{cls}(S)$,
3. $\forall R, S \in SS(L, A)$, *If* $R \subset S$, *then* $\overline{cls}(R) \subset \overline{cls}(S)$,
4. $\forall R \in SS(L, A)$, $\overline{cls}\left(\overline{cls}(R)\right) = \overline{cls}(R)$.

Definition 21 Let $SS(Z, \Sigma)$ and $SS(Y, \Sigma')$ be two soft classes, $f : Z \to Y$ and $g : \Sigma \to \Sigma'$ be mappings and (f, g) be a soft relation on $SS(Z, \Sigma) \times SS(Y, \Sigma')$. Then (f, g) is called a soft function if (f, g) associates each elements of $SS(Z, \Sigma)$ with the unique element of $SS(Y, \Sigma')$. Then a mapping $(f, g) : SS(Z, \Sigma) \to SS(Y, \Sigma')$ is defined as follows:

For all $\sigma_\beta \in SS(Z, \Sigma)$, The images of σ_β is $(f, g)(\sigma_\beta) = f(\sigma)_{g(\beta)}$.

Also, the set $(f, g)(L, A) = \left\{ f(\sigma)_{g(\beta)} : \forall \sigma_\beta \in (L, A) \right\} \subseteq SS(Y, \Sigma')$ for $(L, A) \subseteq SS(Z, \Sigma)$ is called the image of the set (L, A) under (f, g).

Definition 22 Let $SS(Z, \Sigma)$ and $SS(Y, \Sigma')$ be two soft classes and $f : Z \to Y$ and $g : \Sigma \to \Sigma'$ be mappings. $(f, g)^{-1} : SS(Y, \Sigma') \to SS(Z, \Sigma)$ is defined as follows:

For all $\psi_{\beta'} \in SS(Y, \Sigma')$, The inverse image of $\psi_{\beta'}$ is $(f, g)^{-1}(\psi_{\beta'}) = f^{-1}(\psi)_{g^{-1}(\beta')}$.

Also, the set $(f, g)^{-1}(S, B) = \left\{ f^{-1}(\psi)_{g^{-1}(\beta')} : \forall \psi_{\beta'} \in (S, B) \right\} \subseteq SS(Z, \Sigma)$ for $(S, B) \subseteq SS(Y, \Sigma')$ is called the inverse image of the set (S, B).

Example 8 Let $Z = \{\sigma_1, \sigma_2, \sigma_3\}$, $Y = \{\psi_1, \psi_2, \psi_3\}$, $\Sigma = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ and $\Sigma' = \left\{\beta'_1, \beta'_2, \beta'_3\right\}$. Let $f : Z \to Y$ and $g : \Sigma \to \Sigma'$ be mappings defined as

$$f(\sigma_1) = \psi_3 \;\; f(\sigma_2) = \psi_2 \;\; f(\sigma_3) = \psi_2$$
$$g(\beta_1) = \beta'_1 \;\; g(\beta_2) = \beta'_1 \;\; g(\beta_3) = \beta'_3 \;\; g(\beta_4) = \beta'_2$$

We also write (f, g) as below;

$$(f, g)(\sigma_{1_{\beta_1}}) = \psi_{3_{\beta'_1}} \;\; (f, g)(\sigma_{1_{\beta_2}}) = \psi_{3_{\beta'_1}} \;\; (f, g)(\sigma_{1_{\beta_3}}) = \psi_{3_{\beta'_3}} \;\; (f, g)(\sigma_{1_{\beta_4}}) = \psi_{3_{\beta'_2}}$$
$$(f, g)(\sigma_{2_{\beta_1}}) = \psi_{2_{\beta'_1}} \;\; (f, g)(\sigma_{2_{\beta_2}}) = \psi_{2_{\beta'_1}} \;\; (f, g)(\sigma_{1_{\beta_3}}) = \psi_{2_{\beta'_3}} \;\; (f, g)(\sigma_{2_{\beta_4}}) = \psi_{2_{\beta'_2}}$$
$$(f, g)(\sigma_{3_{\beta_1}}) = \psi_{2_{\beta'_1}} \;\; (f, g)(\sigma_{3_{\beta_2}}) = \psi_{2_{\beta'_1}} \;\; (f, g)(\sigma_{3_{\beta_3}}) = \psi_{2_{\beta'_3}} \;\; (f, g)(\sigma_{3_{\beta_4}}) = \psi_{2_{\beta'_2}}$$

We have define (L, A) and (S, B) in (Z, Σ) and (Y, Σ'), respectively, as follows,

$$(L, A) = \{(\beta_1, (\sigma_1, \sigma_2, \sigma_3)), (\beta_2, (\sigma_1, \sigma_2, \sigma_3)), (\beta_4, (\sigma_1, \sigma_2, \sigma_3))\},$$
$$(S, B) = \left\{(\beta'_1, (\psi_1, \psi_2, \psi_3,)), (\beta'_2, (\psi_1, \psi_2, \psi_3)), (\beta'_3, (\psi_1, \psi_2, \psi_3))\right\}$$

We can also write the soft sets (L, A) and (S, B) pointwise as follows.

$$(L, A) = \left\{ \begin{matrix} \sigma_{1\beta_1}, \sigma_{2\beta_1}, \sigma_{3\beta_1}, \sigma_{1\beta_2}, \sigma_{2\beta_2}, \sigma_{3\beta_2} \\ \sigma_{1\beta_4}, \sigma_{2\beta_4}, \sigma_{3\beta_4} \end{matrix} \right\},$$
$$(S, B) = \left\{ \begin{matrix} \psi_{1\beta'_1}, \psi_{2\beta'_1}, \psi_{3\beta'_1}, \psi_{1\beta'_2}, \psi_{2\beta'_2}, \psi_{3\beta'_2} \\ \psi_{1\beta'_3}, \psi_{2\beta'_3}, \psi_{3\beta'_3} \end{matrix} \right\}$$

Then the soft image of (L, A) under $(f, g) : SS(Z, \Sigma) \to SS(Y, \Sigma')$ is obtained as

$$\begin{aligned}(f, g)((L, A)) &= \left\{\psi_{3_{\beta'_1}}, \psi_{2_{\beta'_1}}, \psi_{2_{\beta'_1}}, \psi_{3_{\beta'_1}}, \psi_{2_{\beta'_1}}, \psi_{2_{\beta'_1}}, \psi_{3_{\beta'_2}}, \psi_{2_{\beta'_2}}, \psi_{2_{\beta'_2}}\right\}, \\ &= \left\{\psi_{3_{\beta'_1}}, \psi_{2_{\beta'_1}}, \psi_{3_{\beta'_2}}, \psi_{2_{\beta'_2}}\right\}\end{aligned}$$

Then, we have

$$(f, g)((L, A)) = \left\{(\beta'_1, (\psi_2, \psi_3)), (\beta'_2, (\psi_2, \psi_3))\right\}$$

Now, the soft inverse image of (S, B) under $(f, g)^{-1} : SS(Y, \Sigma') \to SS(Z, \Sigma)$ is obtained as

$$\begin{aligned}(f, g)^{-1}\left(\psi_{1_{\beta'_1}}\right) &= \emptyset, \\ (f, g)^{-1}\left(\psi_{2_{\beta'_1}}\right) &= \left\{\sigma_{2_{\beta_1}}, \sigma_{2_{\beta_2}}, \sigma_{3_{\beta_1}}, \sigma_{3_{\beta_2}}\right\}, \\ (f, g)^{-1}\left(\psi_{3_{\beta'_1}}\right) &= \left\{\sigma_{1_{\beta_1}}, \sigma_{1_{\beta_2}}\right\}, \\ (f, g)^{-1}\left(\psi_{1_{\beta'_2}}\right) &= \emptyset, \\ (f, g)^{-1}\left(\psi_{2_{\beta'_2}}\right) &= \left\{\sigma_{2_{\beta_4}}, \sigma_{3_{\beta_4}}\right\}, \\ (f, g)^{-1}\left(\psi_{3_{\beta'_2}}\right) &= \left\{\sigma_{1\beta_4}\right\}, \\ (f, g)^{-1}\left(\psi_{1_{\beta'_3}}\right) &= \emptyset, \\ (f, g)^{-1}\left(\psi_{2_{\beta'_3}}\right) &= \emptyset, \\ (f, g)^{-1}\left(\psi_{3_{\beta'_3}}\right) &= \emptyset\end{aligned}$$

Then;

$$(f, g)^{-1}((S, B)) = \left\{ \begin{matrix} \sigma_{1_{\beta_1}}, \sigma_{2_{\beta_1}}, \sigma_{3_{\beta_1}}, \sigma_{1_{\beta_2}}, \sigma_{2_{\beta_2}}, \sigma_{3_{\beta_2}} \\ \sigma_{1_{\beta_4}}, \sigma_{2_{\beta_4}}, \sigma_{3_{\beta_4}} \end{matrix} \right\}.$$

Theorem 7 *Let $(f, g) : SS(Z, \Sigma) \to SS(Y, \Sigma')$ be a soft function. Then for soft sets $(L, A) \in SS(Z, \Sigma)$, $(S, B) \in SS(Y, \Sigma')$, we have,*

1. $(f, g)(0_{(Z,\Sigma)}) = 0_{(Y,\Sigma')}$
2. $(f, g)(1_{(Z,\Sigma)}) \subseteq 1_{(Y,\Sigma')}$
3. $(f, g)((L, A) \cup (S, B)) = (f, g)((L, A)) \cup (f, g)((S, B))$
4. $(f, g)((L, A) \cap (S, B)) \subseteq (f, g)((L, A)) \cap (f, g)((S, B))$

5. If $(L, A) \subseteq (S, B)$, then $(f, g)((L, A)) \subseteq (f, g)((S, B))$

Proof We only prove (3)-(5)

(3) Suppose that

$(L, A) \cup (S, B) = (N, A \cup B)$

and

$(f, g)((L, A)) \cup (f, g)((S, B)) = (f(L), g(A)) \cup (f(S), g(B)) = (S, g(A) \cup g(B))$.

Then$(f, g)((L, A) \cup (S, B)) = (f(N), g(A \cup B)) = (f(N), g(A) \cup g(B))$. For any $\psi \in Y$ and $\beta' \in g(A) \cup g(B)$, if $f^{-1}(\psi) = \emptyset$, then

$\mu_{S(\beta')}(\psi) = \mu_{f(N)(\beta')}(\psi) = 0$.

Otherwise, we consider the following cases.

Case 1: $\beta' \in g(A) - g(B)$. Then $S(\beta') = f(L)(\beta')$. On the other hand, $\beta' \in g(A) - g(B)$ implies that there does not exist $\beta \in B$ such that $g(\beta) = \beta'$, that is, for any $\beta = g^{-1}(\beta') \cap (A \cup B)$, we have $\beta \in g^{-1}(\beta') \cap (A - B)$. Hence by Definition 21, we have

$$f(N)(\beta')(\psi) = S(\beta')(\psi)$$

Case 2: $\beta' \in g(B) - g(A)$. Analogous to case 1, we have $f(N)(\beta')(\psi) = S(\beta')(\psi)$.

Case 3: $\beta' \in g(A) \cap g(B)$.Then

$$f(N)(\beta')(\psi) = S(\beta')(\psi)$$

Thus, in any case, $f(N)(\beta')(\psi) = S(\beta')(\psi)$. Therefore, $(f, g)((L, A) \cup (S, B)) = (f, g)((L, A)) \cup (f, g)((S, B))$.

(4) Suppose that $(L, A) \cap (S, B) = (N, A \cup B)$ and

$(f, g)((L, A)) \cap (f, g)((S, B)) = (f(L), g(A)) \cap (f(S), g(B)) = (S, g(A) \cup g(B))$.

Then $(f, g)((L, A) \cap (S, B)) = (f(N), g(A \cup B)) = (f(N), g(A) \cup g(B))$. For any $\psi \in Y$ and $\beta' \in g(A \cup B)$, if $f^{-1}(\psi) = \emptyset$, then

$S(\beta')(\psi) = f(N)(\beta')(\psi) = 0$.

Otherwise, we consider the following cases.

Case 1: $\beta' \in g(A) - g(B)$. Then $S(\beta') = f(L)(\beta')$. On the other hand, $\beta' \in g(A) - g(B)$ implies that there does not exist $\beta \in B$ such that $g(\beta) = \beta'$, that is, for any $\beta = g^{-1}(\beta') \cap (A \cup B)$, we have $\beta \in g^{-1}(\beta') \cap (A - B)$. We have

$$f(N)(\beta')(\psi) \subseteq S(\beta')(\psi)$$

Case 2: $\beta' \in g(B) - g(A)$. Analogous to case 1, we have $f(N)(\beta')(\psi) \subseteq S(\beta')(\psi)$.

Case 3: $\beta' \in g(A) \cap g(B)$.Then

$$f(N)(\beta')(\psi) \subseteq S(\beta')(\psi)$$

Therefore $(f, g)((L, A) \cap (S, B)) \subseteq (f, g)((L, A)) \cap (f, g)((S, B))$.

(5) Let $(L, A) \subseteq (S, B)$.Then $A \subseteq B$ and for any $\beta \in A$ and $\sigma \in Z$, we have $f(A) \subseteq f(B)$, we have,

$$f(L)(\beta')(\psi) \subseteq f(S)(\beta')(\psi)$$

Therefore, $f((L, A)) \subseteq f((S, B))$. □

Theorem 8 *Let $(f, g) : SS(Z, \Sigma) \to SS(Y, \Sigma')$ be a soft function. Then for soft sets $(L, A) \in SS(Z, \Sigma)$, $(S, B) \in SS(Y, \Sigma')$, we have,*

1. $(f,g)^{-1}(0_{(Y,\Sigma')}) = 0_{(Z,\Sigma)}$
2. $(f,g)^{-1}(1_{(Y,\Sigma')}) \subseteq 1_{(Z,\Sigma)}$
3. $(f,g)^{-1}((L,A)\widetilde{\cup}(S,B)) = (f,g)^{-1}((L,A))\,\widetilde{\cup}(f,g)^{-1}((S,B))$
4. $(f,g)^{-1}((L,A)\widetilde{\cap}(S,B)) = (f,g)^{-1}((L,A))\,\widetilde{\cap}(f,g)^{-1}((S,B))$
5. *If* $(L,A) \subseteq (S,B)$, *then* $(f,g)^{-1}((L,A)) \subseteq (f,g)^{-1}((S,B))$
6. $(L,A) \subseteq (f,g)^{-1}((f,g)((L,A)))$, $(f,g)\left((f,g)^{-1}((S,B))\right) = (S,B) \cap (f,g)\left(1_{(Z,\Sigma)}\right)$

Proof Straightforward. □

Definition 23 Let $SS(Z,\Sigma)$, $SS(Y,\Sigma')$ be two soft classes, $(L,A) \in SS(Z,\Sigma)$, $(S,B) \in SS(Y,\Sigma')$. Then $(f,g) = (f,g) : SS(Z,\Sigma) \to SS(Y,\Sigma')$ be a soft mapping such that $f : Z \to Y$, $g : \Sigma \to \Sigma'$.

a The soft mapping (f,g) is called a soft injective function if for every $\sigma_{1\beta}, \sigma_{2\beta} \in (L,A)$, $\sigma_{1\beta} \neq \sigma_{2\beta}$ implies $(f,g)\left(\sigma_{1\beta}\right) = (f(\sigma_1), g(\beta)) \neq (f,g)\left(\sigma_{2\beta}\right) = (f(\sigma_2), g(\beta))$.
b The soft mapping (f,g) is called a soft surjective function if there exists a soft point $\sigma_\beta \in (L,A)$, such that $(f,g)(\sigma_\beta) = \psi_{\beta'}^{'}$ for every $\psi_{\beta'} \in (S,B)$.
c The soft mapping (f,g) is called a soft bijective function if (f,g) is both injective and surjective.
d The soft mapping (f,g) is called a soft constant function if $(f,g)(\sigma_\beta) = \psi_{\beta'}$ is provided for $\forall\sigma_\beta \in (L,A)$, $\exists\psi_{\beta'} \in (S,B)$.

Example 9 Let $Z = \{\sigma_1,\sigma_2\}$, $Y = \{\psi_1,\psi_2\}$, $\Sigma = \{\beta_1,\beta_2\}$ and $\Sigma' = \left\{\beta_1',\beta_2'\right\}$. Let $f : Z \to Y$ and $g : \Sigma \to \Sigma'$ be mappings defined as

$$f(\sigma_1) = \psi_1, \quad f(\sigma_2) = \psi_2$$
$$g(\beta_1) = \beta_2', \quad g(\beta_2) = \beta_1'$$

We also write (f,g) as below;

$$(f,g)(\sigma_{1_{\beta_1}}) = \psi_{1_{\beta_2'}} \;\; (f,g)(\sigma_{1_{\beta_2}}) = \psi_{1_{\beta_1'}}$$
$$(f,g)(\sigma_{2_{\beta_1}}) = \psi_{2_{\beta_2'}} \;\; (f,g)(\sigma_{2_{\beta_2}}) = \psi_{2_{\beta_1'}}$$

We have define (L,A) and (S,B) in (Z,Σ) and (Y,Σ'), respectively, as follows,

$$(L,A) = \{(\beta_1,\{\sigma_1,\sigma_2\}),(\beta_2,\{\sigma_1,\sigma_2\})\}$$
$$(S,B) = \left\{(\beta_1',\{\psi_1,\psi_2\}),(\beta_2',\{\psi_1,\psi_2\})\right\}$$

We can also write the soft sets (L,A) and (S,B) pointwise as follows.

$$(L,A) = \left\{\sigma_{1_{\beta_1}},\sigma_{2_{\beta_1}},\sigma_{1_{\beta_2}},\sigma_{2_{\beta_2}}\right\},$$
$$(S,B) = \left\{\psi_{1_{\beta_1'}},\psi_{2_{\beta_1'}},\psi_{1_{\beta_2'}},\psi_{2_{\beta_2'}}\right\}$$

Then the soft image of (L,A) under $(f,g) : (Z,\Sigma) \to (Y,\Sigma')$ is obtained as

$$(f,g)((L,A)) = \left\{\psi_{1_{\beta_2'}},\psi_{1_{\beta_1'}},\psi_{2_{\beta_1'}},\psi_{2_{\beta_2'}}\right\}$$

Then, we have

$$(f,g)((L,A)) = \left\{(\beta_1',\{\psi_1,\psi_2\}),(\beta_2',\{\psi_1,\psi_2\})\right.$$

It is clear that, for any two different points selected from the soft set (L, A), the images of these points under (f, g) mapping are different from each other. Therefore this mapping is soft injective mapping.

For any soft soft point from the selected the soft set (S, B), there exist a soft point $\sigma_\beta \in (L, A)$ such that $(f, g)(\sigma_\beta) = \psi_{\beta'}$. Therefore this mapping is also soft surjective mapping.

Since the soft mapping (f, g) is both injective and surjective, it is bijective.

Definition 24 Let $(f, g) : SS(Z, \Sigma) \to SS(Y, \Sigma')$, $(h, z) : SS(Z, \Sigma) \to SS(X, \Sigma'')$ be soft mappings. Soft mappings (f, g) and (h, z) is called two soft equal mappings if for every $\sigma_\beta \in (Z, \Sigma)$ implies $(f, g)\left(\sigma_\beta\right) = (h, z)(\sigma_\beta)$. It is denoted by $(f, g) = (h, z)$.

Definition 25 Let $(f, g) : SS(Z, \Sigma) \to SS(Y, \Sigma')$, $(h, z) : SS(Y, \Sigma') \to SS(X, \Sigma'')$ be soft mappings. Then the composition of (f, g) and (h, z), denoted by $(h, z)o(f, g) : SS(Z, \Sigma) \to SS(X, \Sigma'')$, is a soft mapping and defined by $((h, z)o(f, g))(\sigma_\beta) = (h, z)\left((f, g)\left(\sigma_\beta\right)\right)$ for $\sigma_\beta \in SS(Z, \Sigma)$.

Proposition 9 *Let $(f, g) : SS(Z, \Sigma) \to SS(Y, \Sigma')$, $(h, z) : SS(Y, \Sigma') \to SS(X, \Sigma'')$ be two soft mappings. We have;*

1. *If the soft mappings (f, g) and (h, z) are two soft injective mappings then the composition of (f, g) and (h, z), $(h, z)o(f, g) : SS(Z, \Sigma) \to SS(X, \Sigma'')$ is also soft injective mappings.*
2. *If the soft mappings (f, g) and (h, z) are two soft suurjective mappings then the composition of (f, g) and (h, z), $(h, z)o(f, g) : SS(Z, \Sigma) \to SS(X, \Sigma'')$ is also soft surjective mappings.*
3. *If the soft mappings (f, g) and (h, z) are two soft bijective mappings then the composition of (f, g) and (h, z), $(h, z)o(f, g) : SS(Z, \Sigma) \to SS(X, \Sigma'')$ is also soft bijective mappings.*

Proof Straightforward. □

4 Decision making method

This section presents an application of soft relations in a decision-making problem.

The problem we are dealing with is as follows.

Let $Z = \{\sigma_1, \sigma_2, \sigma_3\}$ be a set of pants $\sigma_1, \sigma_2, \sigma_3$, $Z' = \left\{\sigma'_1, \sigma'_2, \sigma'_3\right\}$ be a set of shirts $\sigma'_1, \sigma'_2, \sigma'_3$ and $\Sigma = \{\beta_1, \beta_2, \beta_3\}$ be color of cars $\beta_1 = Grey$, $\beta_2 = Blue$, $\beta_3 = Black$ $\Sigma' = \left\{\beta'_1, \beta'_2, \beta'_3\right\}$ be a set of size, $\beta'_1 = S$, $\beta'_2 = M$, $\beta'_3 = L$. Mr. X wants to do a pants-shorts combo. He defines two different choice sets $(L, A) \in SS(Z, \Sigma)$, $(S, B) \in SS(Z', \Sigma')$ as follows.

$$(L, A) = \{(\beta_1, (\sigma_1, \sigma_2, \sigma_3)), (\beta_2, (\sigma_1, \sigma_2, \sigma_3)), (\beta_3, (\sigma_1, \sigma_2, \sigma_3))\},$$
$$(S, B) = \left\{(\beta'_1, \left(\sigma'_1, \sigma'_2, \sigma'_3\right)), (\beta'_2, \left(\sigma'_1, \sigma'_2\right))\right\}.$$

One possible way to select a clothing combinations based on its parameters is by using soft relation and following the algorithm below.

Step 1. Construct two soft sets over Z, (L, A) and (S, B)

Step 2. Calculate soft cartesian product of (L, A) and (S, B)

Step 3. Construct the relation using cartesian product of (L, A) and (S, B)

Step 4. A weight is assigned for each pair of parameters in relation.

Step 5. Present the weighted (w_i) relation in tabular form by computing.

Step 6. The value of each alternative is multiplied by the weight of the parameter and the score (s_i) column is formed.

Step 7. The alternative belonging to the $maxs_i$ value is selected.

The steps in the algorithm are applied as follows, taking into account the above (L, A) and (S, B) soft sets.

The decision sets of the decision maker $((L, A)$ and $(S, B))$ are multiplied and their soft cartesian product sets are found as follows.

$$(L,A)\times(S,B)=\left\{\begin{array}{c}\left[\left(\beta_1,\beta_1'\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_1,\sigma_3'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\\ \left(\sigma_2,\sigma_3'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right),\left(\sigma_3,\sigma_3'\right)\end{array}\right)\right],\\ \left[\left(\beta_1,\beta_2'\right),\left(\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right)\right)\right],\\ \left[\left(\beta_2,\beta_1'\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_1,\sigma_3'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\\ \left(\sigma_2,\sigma_3'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right),\left(\sigma_3,\sigma_3'\right)\end{array}\right)\right],\\ \left[\left(\beta_2,\beta_2'\right),\left(\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right)\right)\right],\\ \left[\left(\beta_3,\beta_1'\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_1,\sigma_3'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\\ \left(\sigma_2,\sigma_3'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right),\left(\sigma_3,\sigma_3'\right)\end{array}\right)\right],\\ \left[\left(\beta_3,\beta_2'\right),\left(\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right)\right)\right]\end{array}\right\}$$

Decision makers have constructed the relation R, which is a subset of the set of cartesian products (L, A) and (S, B), as follows. In such a product set, a partial elimination is made.

$$R=\left\{\begin{array}{c}\left[\left(\beta_1,\beta_1'\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_1,\sigma_3'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\\ \left(\sigma_2,\sigma_3'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right),\left(\sigma_3,\sigma_3'\right)\end{array}\right)\right],\\ \left[\left(\beta_2,\beta_1'\right),\left(\begin{array}{c}\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_1,\sigma_3'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\\ \left(\sigma_2,\sigma_3'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right),\left(\sigma_3,\sigma_3'\right)\end{array}\right)\right],\\ \left[\left(\beta_2,\beta_2'\right),\left(\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right)\right)\right],\\ \left[\left(\beta_3,\beta_2'\right),\left(\left(\sigma_1,\sigma_1'\right),\left(\sigma_1,\sigma_2'\right),\left(\sigma_2,\sigma_1'\right),\left(\sigma_2,\sigma_2'\right),\left(\sigma_3,\sigma_1'\right),\left(\sigma_3,\sigma_2'\right)\right)\right]\end{array}\right\}$$

Assume that decision maker assigns the weight of the parameters as $w_1 = 0,5$, $w_2 = 0,7$, $w_3 = 0,2$, $w_4 = 0,6$. We compute the comparison table as Table 1 following;

It is clear from the statement highlighted in bold in Table-1 that max $S_i = 1,5$ is the highest value. Hence Mr. X will choose $\left(\sigma_3, \sigma_2'\right)$. So So he will choose σ_3 pants and σ_2' shirts.

5 Conclusion

In this paper, we propose a new definition for the soft Cartesian product. Using this definition, we then redefine the terms "soft relation" and "soft function". In contrast to previous research in the literature, all the stated concepts involved many dimensions and alternatives. We talked about the ideas of soft image and inverse image of the given set and looked at many

Table 1 The tabular form of weighted parametrized relation

	$(\beta_1, \beta_1')\,(w_1 = 0,5)$	$(\beta_2, \beta_1')\,(w_2 = 0,7)$	$(\beta_2, \beta_2')\,(w_3 = 0,2)$	$(\beta_3, \beta_2')\,(w_4 = 0,6)$	$\frac{S_i}{Score}$
(σ_1, σ_1')	1	0	1	0	0,7
(σ_1, σ_2')	1	0	1	1	1,3
(σ_1, σ_3')	0	0	0	0	0
(σ_2, σ_1')	1	0	0	0	0,5
(σ_2, σ_2')	0	1	1	0	0,9
(σ_2, σ_3')	1	0	0	0	0,5
(σ_3, σ_1')	1	0	0	1	1,1
(σ_3, σ_2')	0	1	1	1	**1,5**
(σ_3, σ_3')	1	1	0	0	1,2

bold value highlight largest score in the table

of its properties. The ideas of injective, surjective, bijective and constant functions were also discussed. The work is enriched by the large number of examples provided. The findings of the study are used in the decision making algorithm in the last section to improve the functionality. In future work, the type of relation we have defined can be taken as a basis and many different structures (such as fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, hyperSoft Set, indetermsoft Set, indetermhyperSoft Set, superhypersoft set, treesoft set) can be studied. We hope that the findings will be useful for future studies.

Funding Open access funding provided by the Scientific and Technological Research Council of Türkiye (TÜBİTAK). No funding was received for conducting this study.

Data Availability Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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