Iranian Journal of Science (2026) 50:655–662
https://doi.org/10.1007/s40995-025-01859-1

RESEARCH PAPER

On the Finite Delayed Fractional Differential Equation via Riemann–Liouville Derivative of Non-linear Variable-Order

Mohammed Said Souid[1,2] · Hatira Günerhan[3] · Hallouz Abdelhamid[4] · Hemen Dutta[5]

Received: 9 February 2025 / Accepted: 12 July 2025 / Published online: 31 August 2025
© The Author(s), under exclusive licence to Shiraz University 2025

Abstract

This paper investigates the existence and uniqueness of solutions (EUSS) for a finite delayed problem of fractional differential equations with nonlinear variable order (FDPFDENVO). The analysis employs advanced mathematical tools, including the schauder fixed-point theorem (SFPT) and the banach fixed-point theorem (BFPT), offering a streamlined approach that requires fewer restrictive assumptions. To demonstrate the practical relevance of the theoretical findings, the study includes an applied example. The results hold significant value for understanding complex dynamical systems with time delays, with potential applications in engineering, economics, and medicine. Additionally, the findings contribute to refining numerical methods for solving delay equations.

Keywords Derivatives and integrals of variable-order · Fixed point theorem · Initial value problem · Finite delayed problem

Mathematics Subject Classification 26A33 · 34A08 · 34K37

Abbreviations

AAT	Arzela Ascoli theorem
BS	Banach space
BFPT	Banach fixed point theorem
CF	Continuous functio
CC	Complitly continuous
CCBNE	Convex, closed, bounded and nonempty
CS	Convex Subset
CO	Continuous
EC	Equicontinuous
LM	Lebesgue measurable
FP	Fixed point
RC	Relatively compact
RLDVO	Riemann Liouville derivative of variable order
RLFDVO	Riemann Liouville fractional derivative of variable order
RLDCO	Riemann Liouville derivative of constant order
RLFIVO	Riemann Liouville fractional integral of variable order
RLICO	Riemann Liouville integral of constant order
SFPT	Schauder fixed point fheorem
UB	Uniformly bounded
UFP	Unique fixed point
US	Uniform stability

✉ Hatira Günerhan
gunerhanhatira@gmail.com

Mohammed Said Souid
souimed2008@yahoo.com

Hallouz Abdelhamid
abdelhamid.hallouz@univ-tiaret.dz

Hemen Dutta
hemen_dutta08@rediffmail.com

[1] Department of Mathematics, Saveetha School of Engineering, SIMATS, Chennai, India

[2] Department of Economic Sciences, University of Tiaret, Tiaret, Algeria

[3] Department of Mathematics, Faculty of Education, Kafkas University, Kars, Turkey

[4] Laboratory of Informatics and Mathematics, Department of Mathematics, University of Tiaret, 14005 Tiaret, Algeria

[5] Department of Mathematics, Gauhati University, Guwahati, Assam 781014, India

1 Introduction

Differential equations with fractional-order derivatives can be used to Compared to differential equations of integer order, they may simulate systems with limitless memory,

which is advantageous. Fractional-order derivatives are known to better represent the memory and heredity characteristics of specific materials and processes. Moreover, the model has additional degrees of freedom thanks to fractional-order derivatives and integrals. Consequently, differential equations with fractional derivatives have been intensively investigated as a means of accurately defining real systems investigated in many scientific, technical, and medical domains in recent decades. Baleanu et al. (2012); Magin (2006); Kilbas et al. (2006); Petráš (2011); Samko et al. (1993); Stamova and Stamov (2017); Tarasov (2015).

The fundamental idea is to substitute a function $\gamma(\cdot)$ for the constant γ in order to convert constant-order fractional calculus to variable-order fractional calculus. Samko (2013); Samko and Ross (1993); Valerio and Costa (2011); Ahmad et al. (2023); Imtiaz et al. (2025, 2025); Ait Mohammed et al. (2024); Zibar et al. (2025). This advancement leads to the more appropriate use of fractional differential equations with variable order derivatives in the description of many processes and natural phenomena. The theory of these equations has thus been the subject of extensive research (Almeida et al. 2019; Benkerrouche et al. 2022; Garrappa et al. 2021; Odzijewicz et al. 2013; Zhang and Hu 2020; Zhang et al. 2018), as well as recently investigated applications (Patnaik et al. 2020; Lu et al. 2019; Sun et al. 2009; Sweilam and AL-Mekhlafi SM, Alshomrani AS, Baleanu D, 2020).

The authors in Benkerrouche et al. (2021, 2021, 2022, 2021, 2022); Bouazza et al. (2021, 2022); Refice et al. (2021); Rezapour et al. (2021) have made significant contributions in this field in the past few years. As we can see in the literature, there is a large body of work on the existence of different classes of FDEs with varied order of fractional derivatives (Rekhviashvili et al. 2019; Agarwal et al. 2021, 2021; Butt et al. 2024; Kıymaz et al. 2017; Singh et al. 2018).In this work, we investigate the following problem for fractional differential equations (FDE) with nonlinear variable order (NVO) derivatives of Riemann–Liouville type, motivated by the previous related studies:

$$\begin{cases} D_{0^+}^{\Phi(t,u(t))}u(t) = \Upsilon(t,u_t), & t \in \underline{\Delta}_1 := [0,M], \ 0<M<\infty, \quad (A) \\ u(t) = \zeta(t), & t \in \underline{\Delta}_2 := [-r,0], \quad (B) \end{cases} \tag{FDPFDENVO}$$

where $D_{0^+}^{\Phi(t,u(t))}$ stands for the RLFDVO$\Phi(t,u(t))$, $r>0$,Υ is a given function, and Φ satisfies $0<\Phi_* \le \Phi(t,u(t)) \le \Phi^* <1$. $\Upsilon : \underline{\Delta}_1 \times L^1(\underline{\Delta}_2) \to \mathbb{R}$ is a given function and $\zeta \in L^1(\underline{\Delta}_2)$ with $\zeta(0)=0$. For any function u defined on $[-r,M]$ and any $t \in \underline{\Delta}_1$, we denote by u_t the elements of $L^1([-r,0])$ defined by

$$u_t(s) = u(t+s), \ \ s \in [-r,0].$$

Here $u_t(.)$ represents the history of the state from time $t-r$ up to the present time t. We denote $\underline{\Delta} = \underline{\Delta}_1 \cup \underline{\Delta}_2 = [-r,M]$.

The differential equations with nonlinear (VOFD) solutions are more adequate models of various problems investigated in science and engineering. But the results devoted to such equations are rare. Thus, the development of the area imposes the establishment of new existence results.

The following is how the paper is organized: Section 2 contains certain notations, definitions, and lemmas. Section 3 is devoted to new existence results for DFDENVO solutions. The BFPT and SFPT are applied to get the existence result. Section 4 examples that prove Section 3. Finally, Section 5 has concluding observations and future works.

2 Preliminaries

This section deals with preliminaru rasults and some notations and difinitions The Notation used in this work are defined in Appendix A (page 10-11)

The symbol $L^1(\underline{\Delta},\mathbb{R})$ or $L^1(\underline{\Delta})$ simply represents the BS of all functions $v : \underline{\Delta} \to \mathbb{R}$ which are LM such that

$$\int_{-r}^{M} |v(s)|ds < \infty$$

with the norm

$$\|v\|_{L^1(\underline{\Delta})} =: \int_{-r}^{M} |v(s)|ds.$$

Remark 2.1 *The following observations are made to facilitate our research in the next section:*

- *if* $0<M\le 1$ *then* $\begin{cases} M^{\Phi(s,u(s))-1} & \le M^{\Phi_*-1} \\ M^{-\Phi(s,u(s))} & \le M^{-\Phi^*} \end{cases}$
- *if* $1<M$ *then* $\begin{cases} M^{\Phi(s,u(s))-1} & \le 1 \\ M^{-\Phi(s,u(s))} & \le 1 \end{cases}$

  Set $\theta^* = \max\{1, M^{\Phi_*-1}, M^{-\Phi^*}\}$.
- The function $\Gamma(\Phi(t,u(t))) \in L^\infty(\underline{\Delta})$, hence we can set:

$$M_\varphi = ess.\sup\left\{\left|\frac{1}{\Gamma(\Phi(t,\varphi(t)))}\right| | t \in \underline{\Delta}_1\right\}.$$

Let us consider two CF $\Phi : \underline{\Delta}_1 \times \mathbb{R} \to (0,\Phi_*]$ and $\Lambda : \underline{\Delta}_1 \times \mathbb{R} \to (0,\theta^*]$.

Definition 2.1 (Samko (2013); Samko and Ross (1993); Valerio and Costa (2011)) The left RLFIVO Φ, where $\Phi = \Phi(t,\varphi(t))$ for a function $\varphi(t)$ is

$$I_{\sigma_1^+}^{\Phi(t,\varphi(t))}\varphi(t)=\int_{\sigma_1}^{t}\frac{(t-s)^{\Phi(s,\varphi(s))-1}}{\Gamma(\Phi(s,\varphi(s)))}\varphi(s)ds,\quad t>\sigma_1, \tag{1}$$

where $\Gamma(.)$ is the standard Gamma function and $a_1 \in \underline{\Delta}$.

If $\Phi(t,u(t))$ is a constant Φ, then (1) will be reduced to

$$I_{\sigma_1^+}^{\Phi}\varphi(t)=\frac{1}{\Gamma(\Phi)}\int_{\sigma_1^+}^{t}(t-s)^{\Phi-1}\varphi(s)ds,\quad t>\sigma_1, \tag{2}$$

which is the classical RLICO Magin (2006); Kilbas et al. (2006); Samko et al. (1993).

Definition 2.2 (Samko (2013); Samko and Ross (1993); Valerio and Costa (2011)) The left RLFDVO Λ, where $\Lambda=\Lambda(t,\varphi(t))$ for a function $\varphi(t)$ is

$$\begin{aligned}D_{\sigma_1^+}^{\Lambda(t,\varphi(t))}\varphi(t)&=\frac{d}{dt}I_{\sigma_1^+}^{1-\Lambda(t,\varphi(t))}\\ \varphi(t)&=\frac{d}{dt}\int_{\sigma_1}^{t}\frac{(t-s)^{-\Lambda(s,\varphi(s))}}{\Gamma(1-\Lambda(s,\varphi(s)))}\varphi(s)ds,\quad t>\sigma_1.\end{aligned} \tag{3}$$

If $\Lambda(t,u(t))$ is a constant Λ, then (3) will be

$$D_{\sigma_1^+}^{\Lambda}\varphi(t)=\frac{d}{dt}I_{\sigma_1^+}^{1-\Lambda}\varphi(t)=\frac{d}{dt}\int_{\sigma_1}^{t}\frac{(t-s)^{-\Lambda}}{\Gamma(1-\Lambda)}\varphi(s)ds,\quad t>\sigma_1, \tag{4}$$

which is the classical RLDCO Λ Magin (2006); Kilbas et al. (2006); Samko et al. (1993).

We refer to for further information on the characteristics of classical fractional derivatives and integrals of constant order (Magin 2006; Kilbas et al. 2006; Samko et al. 1993), see also fractional integrals and derivatives of variable order in Almeida et al. (2019); Samko (2013); Samko and Ross (1993).

Remark 2.2 *It is well known that the semigroupe property does not hold in the case of the functions* $\Phi(t,u(t))$, $\Lambda(t,u(t))$, *i.e.,*

$$I_{\sigma_1^+}^{\Phi(t,\varphi(t))}I_{\sigma_1^+}^{\Lambda(t,\varphi(t))}\varphi(t)\neq I_{\sigma_1^+}^{\Phi(t,\varphi(t))+\Lambda(t,\varphi(t))}\varphi(t).$$

for counter examples see Zhang (2013); Zhang and Hu (2020); Benkerrouche et al. (2022).

We list some properties of the RLFIVO which will be used in the coming lemmas.

Lemma 2.1 *Hallouz et al. (2023) If* $\Phi:\underline{\Delta}\times\mathbb{R}\to(0,1]$ *is a continuous function, such that* $0<\Phi_*\leq\Phi(t,\varphi(t))\leq\Phi_*<1$, *then* $I_{0^+}^{\Phi(t,\varphi(t))}\varphi(t)\in L^1(\underline{\Delta})$ *for* $\varphi\in L^1(\underline{\Delta})$. *Moreover, we have*:

(i)

$$\|I_{0^+}^{\Phi(t,\varphi(t))}\varphi\|_{L^1(\underline{\Delta})}\leq\frac{(M+r)M_f\theta^*}{\Phi_*}\|\varphi\|_{L^1(\underline{\Delta})}. \tag{5}$$

(ii) For $\varphi,\Psi\in L^p(\underline{\Delta},\mathbb{R})$ we have

$$\begin{aligned}&\|I_{0^+}^{\Phi(t,\varphi(t))}\varphi-I_{0^+}^{\Phi(t,\Psi(t))}\Psi\|_{L^1(\underline{\Delta})}\\ &\leq\frac{4(M+r)B\theta^*}{\Phi_*}\|\varphi-\Psi\|_{L^1(\underline{\Delta})}.\end{aligned} \tag{6}$$

We will also need the following results.

Lemma 2.2 *Kesavan (2019); Haraux and Cazenave (1998) Let J a bounded interval and* $\varphi\in Ł^1(J)$, *then we have*

$$\lim_{h\to0}\frac{1}{h}\int_t^{t+h}\varphi(s)ds=\varphi(t)\ \ a.e \tag{7}$$

Lemma 2.3 (*SFPT*) (*Kilbas wt al. (2006) Kesavan (2004)*) *Let E be a BS and* $\veebar$ *be a CS of E and* $T:\veebar\longrightarrow\veebar$ *is compact, and continuous map. Then F has at least one FP in* $\veebar$.

Lemma 2.4 (*Kesavan (2009)*) *Let* $\veebar$ *be a Banach space and* $T:\veebar\longrightarrow\veebar$ *be a mapping such that for some* $n\in\mathbb{N}$, T^n *is a contraction. Then T has a UFP in* Υ.

3 Achieved Existence Results

Definition 3.1 We say that $u\in L^1(\underline{\Delta})$ is a solution for (FDPFDENVO) if and only if u verifies (FDPFDENVO(A)) and (FDPFDENVO(B)), at the same time.

Prior to asserting and showing the existence easults in the BS $L^1(\underline{\Delta})$, We shall assert and demonstrate the following technical lemma.

Lemma 3.1 *For any* $u\in L^1(\underline{\Delta})$, *equation (FDPFDENVO(A)) take the following equivalent formula*

$$\begin{aligned}I_{0^+}^{1-\Phi(t,u(t))}u(t)&=\int_0^t\frac{(t-s)^{-\Phi(s,u(s))}}{\Gamma(1-\Phi(s,u(s)))}u(s)ds\\ &=\int_0^t\Upsilon(s,u_s)ds,t\in\underline{\Delta}_1.\end{aligned} \tag{8}$$

Proof 1 *Let* $u\in L^p(\underline{\Delta},\mathbb{R})$, *then from difinition(2.2) we can write (FDPFDENVO(A)) simply as*

$$D_{0^+}^{\Phi(t,u(t))}u(t)=\frac{d}{dt}\int_0^t\frac{(t-s)^{-\Phi(s,u(s))}}{\Gamma(1-\Phi(s,u(s)))}u(s)ds=\Upsilon(t,u_t). \tag{9}$$

Now we can integrate (9) from [0, t], to get

$$\int_0^t \frac{(t-s)^{-\Phi(s,u(s))}}{\Gamma(1-\Phi(s,u(s)))}u(s)ds = c_0 + \int_0^t \Upsilon(s,u_s)ds. \quad (10)$$

Inserting $t = 0$in (10) gives $c_0 = 0$, which is Eq(8).

Conversely, let us consider (8), by the considiration in lemma(2.1), we can take its derivative which shows that

$$\frac{d}{dt}\int_0^t \frac{(t-s)^{-\Phi(s,u(s))}}{\Gamma(1-\Phi(s,u(s)))}u(s)ds = \Upsilon(t,u_t), \quad (11)$$

and again difinition(2.2) gives (FDPFDENVO(A)) and conclude the proof.

The following assumptions are critical to our analysis:

(A1) $\Phi : \underline{\Delta}_1 \times \mathbb{R} \to (0, \Phi_*]$ is a CF.

(A2) $\Upsilon : \underline{\Delta}_1 \times L^1(\underline{\Delta}_2) \to \mathbb{R}$ is a CF with respect to its first variable and such that:

$$|\Upsilon(t,\widetilde{\varphi}_1) - \Upsilon(t,\widetilde{\varphi}_2)| \le \tau||\widetilde{\varphi}_1 - \widetilde{\varphi}_2||_{L^1(\underline{\Delta}_2)}, \ \forall \ \widetilde{\varphi}_1, \widetilde{\varphi}_2 \in L^1(\underline{\Delta}_2),$$

$t \in \underline{\Delta}_1$ and $\tau > 0$.

(A3) $\Upsilon : \underline{\Delta}_1 \times L^1(\underline{\Delta}_2) \to \mathbb{R}$ is a CF and such that:

$$|\Upsilon(t,\widetilde{\varphi})| \le |a(t)| + \tau||\widetilde{\varphi}||_{L^1(\underline{\Delta}_2)}, \ \forall \ \widetilde{\varphi} \in L^1(\underline{\Delta}_2), \quad (12)$$

$t \in \underline{\Delta}_1$ and $\tau > 0$, $a \in L^1(\underline{\Delta}_1)$.

In this first existence and uniqueness result we are going to use lemma (2.4).

Theorem 3.1 *If we consider* (A1)-(A2), then (FDPFDENVO) has a unique solution in the BS $L^1(\underline{\Delta})$.

Proof 2 *Let v, u elements in* $L^1(\underline{\Delta})$ *, and the operator*

$$\widetilde{\mathcal{X}}_\varepsilon : L^1(\underline{\Delta}) \to L^1(\underline{\Delta}),$$

defined as follow

$$\left(\widetilde{\mathcal{X}}_\varepsilon u\right)(t) = \begin{cases} \zeta(t), & t \in \underline{\Delta}_2, \\ \varepsilon u(t) + I_{0^+}^{1-\Phi(t,u(t))}u(t) - \int_0^t \Upsilon(s,u_s)ds, & t \in \underline{\Delta}_1, \end{cases} \quad (13)$$

Where $0 < \varepsilon < 1$. by using the assumption (A2), we have

$$\left|\int_0^t \Upsilon(s,v_s) - \Upsilon(s,u_s)ds\right| \le \tau\int_0^M ||v_s - u_s||_{L^1(\underline{\Delta}_2)}ds \le M\tau||v-u||_{L^1(\underline{\Delta})}. \quad (14)$$

Using Eq.(14) we get

$$\begin{aligned} \left|\left(\widetilde{\mathcal{X}}_\varepsilon v\right)(t) - \left(\widetilde{\mathcal{X}}_\varepsilon u\right)(t)\right| &\le \varepsilon|v(t) \\ -u(t)| + |I_{0^+}^{1-\Phi(t,v(t))}v(t) - I_{0^+}^{1-\Phi(t,u(t))}u(t))| &\\ +\left|\int_0^t \Upsilon(s,v(s)) - \Upsilon(s,u(s))ds\right| &\\ \le \varepsilon|u(t)-v(t)| + \left|I_{0^+}^{1-\Phi(t,v(t))}v(t) - I_{0^+}^{1-\Phi(t,u(t))}u(t))\right| &\\ +M\tau||v-u||_{L^1(\underline{\Delta})}. & \end{aligned} \quad (15)$$

By integer both sides of Eq.(15) on $\underline{\Delta}$, we get

$$\begin{aligned} \left|\left|\widetilde{\mathcal{X}}_\varepsilon v - \widetilde{\mathcal{X}}_\varepsilon u\right|\right|_{L^1(\underline{\Delta})} &\le \varepsilon||v-u||_{L^1(\underline{\Delta})} \\ + \left|\left|I_{0^+}^{1-\Phi(t,v(t))}v - I_{0^+}^{1-\Phi(t,u(t))}u\right|\right|_{L^1(\underline{\Delta})} &+ M^2\tau||v-u||_{L^1(\underline{\Delta})}. \end{aligned} \quad (16)$$

Using Eq.(6) from lemma 2.1 , we get

$$\left|\left|\widetilde{\mathcal{X}}_\varepsilon v - \widetilde{\mathcal{X}}_\varepsilon u\right|\right|_{L^1(\underline{\Delta})} \le \left(\varepsilon + \frac{4(M+r)B\theta^*}{1-\Phi^*} + M^2\tau\right) ||v-u||_{L^1(\underline{\Delta})}. \quad (17)$$

Set $\theta = \varepsilon + \frac{4(M+r)B\theta^*}{1-\Phi^*} + M^2\tau$. so that we have

$$\left|\left|\widetilde{\mathcal{X}}_\varepsilon v - \widetilde{\mathcal{X}}_\varepsilon u\right|\right|_{L^1(\underline{\Delta})} \le \theta||v-u||_{L^1(\underline{\Delta})},$$

by induction, we can prove that

$$\left|\left|\widetilde{\mathcal{X}}_\varepsilon^n v - \widetilde{\mathcal{X}}_\varepsilon^n u\right|\right|_{L^1(\underline{\Delta})} \le \frac{\theta^n}{n!}||v-u||_{L^1(\underline{\Delta})},$$

where $\widetilde{\mathcal{X}}_\varepsilon^n = \widetilde{\mathcal{X}}_\varepsilon \circ \widetilde{\mathcal{X}}_\varepsilon \circ \widetilde{\mathcal{X}}_\varepsilon \circ \widetilde{\mathcal{X}}_\varepsilon \circ \ldots\ldots \circ \widetilde{\mathcal{X}}_\varepsilon$ "n times".

We have $\lim_{n\to\infty}\frac{\sigma^n}{n!} = 0$. *Then, for sufficiently large n, we get* $\frac{\sigma^n}{n!} < 1$.

Now applying lemma 2.4, there exist a unique $u \in L^1(\underline{\Delta})$ *such that*

$$\left(\widetilde{\mathcal{X}}_\varepsilon u\right)(t) = u(t). \quad (18)$$

If we consider $\varepsilon = \frac{1}{2}$ and $t \in \underline{\Delta}_1$ we deduce

$$u(0) = \zeta(0) = 0.$$

Take the limit as $\varepsilon \to 1$ in the interval $\underline{\Delta}_1$, we reach that

$$I_{0^+}^{1-\Phi(t,u(t))}u(t) = \int_0^t \Upsilon(s,u_s)ds, t \in \underline{\Delta}_1. \quad (19)$$

At this point Lemma(3.1) going to be crucial and give

$$\begin{cases} D_{0^+}^{\Phi(t,u(t))}u(t) = \Upsilon(t,u_t), & t \in \underline{\Delta}_1, \\ u(t) = \zeta(t) & t \in \underline{\Delta}_2. \end{cases} \quad (20)$$

This concludes our proof.

In this scond existence result we are going to use Lemma (2.3).

Theorem 3.2 *If we consider* (A1)-(A3) and the condition

$$\widetilde{K} = \frac{(M+r)M_x\theta^*}{1-\Phi^*} + M^2\tau < 1. \tag{21}$$

Then the (FDPFDENVO) has at least a solution in $L^1(\underline{\Delta})$.

Proof 3 *Let v, u elements in* $L^1(\underline{\Delta})$, $t \in \underline{\Delta}$ *and consider the operator*

$$\widetilde{\mathcal{X}}_{\varepsilon_n} : L^1(\underline{\Delta}) \to L^1(\underline{\Delta}),$$

where

$$\left(\widetilde{\mathcal{X}}_{\varepsilon_n}u\right)(t) = \begin{cases} \zeta(t), & t \in \underline{\Delta}_2, \\ \varepsilon_n u(t) + I_{0^+}^{1-\Phi(t,u(t))}u(t) - \displaystyle\int_0^t \Upsilon(s,u_s)ds, & t \in \underline{\Delta}_1. \end{cases} \tag{22}$$

and $\varepsilon_n = \dfrac{2a}{n+a}$ and $a > 0$ a real number made to be fixed after in the proof.For $t \in \underline{\Delta}_2$

$$\left|\left(\widetilde{\mathcal{X}}_{\varepsilon_n}v\right)(t)\right| \le |\zeta(t)|. \tag{23}$$

From (A3) we have that

$$\begin{aligned} \left|\int_0^t \Upsilon(s,v_s)ds\right| &\le ||a||_{L^1(\underline{\Delta}_1)} + \tau\int_0^M ||v_s||_{L^1(\underline{\Delta}_2)}ds \\ &\le ||a||_{L^1(\underline{\Delta}_1)} + M\tau||v||_{L^1(\underline{\Delta})}. \end{aligned} \tag{24}$$

For $t \in \underline{\Delta}_2$ and by mean of Eq.(24) we get

$$\begin{aligned} \left|\left(\widetilde{\mathcal{X}}_{\varepsilon_n}v\right)(t)\right| &\le \varepsilon_n|v(t)| + \left|I_{0^+}^{1-\Phi(t,v(t))}v(t)\right| \\ &\quad + \left|\int_0^t \Upsilon(s,v(s))ds\right| \\ &\le \varepsilon_n|v(t)| + \left|I_{0^+}^{1-\Phi(t,v(t))}v(t)\right| + ||a||_{L^1(\underline{\Delta}_1)} \\ &\quad + M\tau||v||_{L^1(\underline{\Delta})}. \end{aligned} \tag{25}$$

Combining Eq.(25) and Eq.(23) we get

$$\begin{aligned} &\left|\left|\widetilde{\mathcal{X}}_{\varepsilon_n}v\right|\right|_{L^1(\underline{\Delta})} \le \varepsilon_n||v||_{L^1(\underline{\Delta})} \\ &+ \left|\left|I_{0^+}^{1-\Phi(t,v(t))}v\right|\right|_{L^1(\underline{\Delta})} + M||a||_{L^1(\underline{\Delta}_1)} + M^2\tau||v||_{L^1(\underline{\Delta})} \\ &+ ||\zeta||_{L^1(\underline{\Delta}_2)}. \end{aligned} \tag{26}$$

Using Eq.(6) from Lemma 4, we get

$$\begin{aligned} \left|\left|\widetilde{\mathcal{X}}_{\varepsilon_n}v\right|\right|_{L^1(\underline{\Delta})} &\le \left(\varepsilon_n + \frac{(M+r)M_x\theta^*}{1-\Phi^*} + M^2\tau\right) \\ &||v||_{L^1(\underline{\Delta})} + M||a||_{L^1(\underline{\Delta}_1)} + ||\zeta||_{L^1(\underline{\Delta}_2)}. \end{aligned} \tag{27}$$

There is a suffitiently large integer N_0 for wich

$$\varepsilon_n < 1 - \widetilde{K} \ \ \forall n \ge N_0. \tag{28}$$

And the choice of N_0 is independent from the choice of $d > 0$ fixed in the bigining of the proof Set

$$R = \frac{||\zeta||_{L^1(\underline{\Delta}_2)} + M||a||_{L^1(\underline{\Delta}_1)}}{1 - \left(\varepsilon_n + \widetilde{K}\right)}. \tag{29}$$

Consider

$$B_R = \left\{v \in L^1(\underline{\Delta}) \ \mid \ ||v||_{L^1(\underline{\Delta})} \le R\right\}. \tag{30}$$

It is well known that B_R is a closed non-empty bounded convex set. We will proove that $\widetilde{\mathcal{X}}_{\varepsilon_n} : B_R \to B_R$ is continuous and compact in few steps. **step1:** The fact that $\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R) \subset B_R$ is bay construction. **step2:** $\widetilde{\mathcal{X}}_{\varepsilon_n}$ is continuous, let $(v_n)_{n\ge 0} \subset B_R$, and $v \in B_R$ such that $v_n \underset{n\to+\infty}{\longrightarrow} v$. We have to state and prove the following lemma

Theorem 3.3 *We have the following convergence*

$$\left|\left|I_{0^+}^{1-\Phi(t,v_n(t))}v_n - I_{0^+}^{1-\Phi(t,v(t))}v\right|\right|_{L^1(\underline{\Delta})} \underset{n\to+\infty}{\longrightarrow} 0. \tag{31}$$

$$\int_0^M \left|\Upsilon(s,v_{s,n}) - \Upsilon(s,v_s)\right|ds \underset{n\to+\infty}{\longrightarrow} 0. \tag{32}$$

Proof 4 *For Eq.(31) usig Eq.(6) we get*

$$\begin{aligned} &\left|\left|I_{0^+}^{1-\Phi(t,v_n(t))}v_n - I_{0^+}^{1-\Phi(t,v(t))}v\right|\right|_{L^1(\underline{\Delta})} \le \frac{4(M+r)B\theta^*}{1-\Phi^*} \\ &||v_n - v||_{L^1(\underline{\Delta})} \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned} \tag{33}$$

For Eq.(32) we have $\left|\Upsilon(s,v_{n,s}) - \Upsilon(s,v_s)\right| \underset{n\to+\infty}{\longrightarrow} 0$ and

$$\left|\Upsilon(s,v_{s,n}) - \Upsilon(s,v_s)\right| \le 2|a(t)| + c\tau||v||_{L^1(\underline{\Delta})}.$$

as a cosequence of (A3) and this insures that the DCT is applied and this is exactly (32).

We have

$$\begin{aligned} &\left|\left|\widetilde{\mathcal{X}}_{\varepsilon_n}v_n - \widetilde{\mathcal{X}}_{\varepsilon_n}v\right|\right|_{L^1(\underline{\Delta})} \le \varepsilon_n||v_n - v||_{L^1(\underline{\Delta})} \\ &\quad + \left|\left|\int_0^M \left|\Upsilon(s,v_{s,n}) - \Upsilon(s,v_s)\right|ds\right|\right|_{L^1(\underline{\Delta})} \\ &+ \left|\left|I_{0^+}^{1-\Phi(t,v_n(t))}v_n - I_{0^+}^{1-\Phi(t,v(t))}v\right|\right|_{L^1(\underline{\Delta})} \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned} \tag{34}$$

Which means that $\widetilde{\mathcal{X}}_{\varepsilon_n}$ is continuous.

step3: $\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R)$ is uniformly bounded by construction, it rmains to prove that

$$\left\|\frac{1}{h}\int_t^{t+h}\left(\widetilde{\mathcal{X}}_{\varepsilon_n}v\right)(s)ds-\widetilde{\mathcal{X}}_{\varepsilon_n}v\right\|_{L^1(\underline{\Delta})}\underset{h\to 0}{\longrightarrow}0 \ \ Uniformly. \tag{35}$$

We have

$$\begin{aligned}
&\left\|\frac{1}{h}\int_t^{t+h}\left(\widetilde{\mathcal{X}}_{\varepsilon_n}v\right)(s)ds-\widetilde{\mathcal{X}}_{\varepsilon_n}v\right\|_{L^1(\underline{\Delta})}\\
&\leq \varepsilon_n\int_0^M\left|\frac{1}{h}\int_t^{t+h}v(s)-v(t)ds\right|dt\\
&\quad+\int_{-r}^0\left|\frac{1}{h}\int_t^{t+h}\zeta(s)-\zeta(t)ds\right|dt\\
&+\int_0^M\left|\frac{1}{h}\int_t^{t+h}I_{0^+}^{1-\Phi(s,v(s))}v(s)-I_{0^+}^{1-\Phi(t,v(t))}v(t)ds\right|dt\\
&+\int_0^M\left|\frac{1}{h}\int_t^{t+h}\int_0^s\Upsilon(\rho,v_\rho)d\rho-\int_0^t\Upsilon(\sigma,v_\sigma)d\sigma ds\right|dt.
\end{aligned} \tag{36}$$

Since $v, I_{0^+}^{1-\Phi(s,v(s))}v, \int_0^s\Upsilon(\rho,v_\rho)d\rho\in L^1(\underline{\Delta}_1)$, and $\zeta\in L^1(\underline{\Delta}_1)$ and as a consequence of Lemma 2.2 and another application of DCT, we conclude that Eq.(35) is satisfied. By Kolmogorov's theorem, $\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R)$ is relatively compact, and hence by Lemma (2.3), $\widetilde{\mathcal{X}}_{\varepsilon_n}$ has at least one fixed point in B_R.

$$\left(\widetilde{\mathcal{X}}_{\varepsilon_n}u\right)(t)=u(t)\ \ \forall n\geq N_0.$$

At this point we can fixe $a=N_0$, $n=2N_0$ and $t\in\underline{\Delta}_1$ to deduce

$$u(0)=\zeta(0)=0.$$

In the same way choose $n=N_0$ in the interval $\underline{\Delta}_1$, we get

$$I_{0^+}^{1-\Phi(t,u(t))}u(t)=\int_0^t\Upsilon(s,u_s)ds, t\in\underline{\Delta}_1. \tag{37}$$

Using Lemma (3.1), then we can conclude that

$$\begin{cases}D_{0^+}^{\Phi(t,u(t))}u(t)=\Upsilon(t,u_t), & t\in\underline{\Delta}_1.\\ u(t)=\zeta(t) & t\in\underline{\Delta}_2.\end{cases} \tag{38}$$

This concludes our proof.

4 An Approval Example

Example 1: Let us consider the following (FDPFDENVO)

$$\begin{cases}D_{0^+}^{\Phi(t,u(t))}u(t)=\dfrac{e^{at}\|u_t\|_{L^1(\underline{\Delta})}}{3(e^{at}+e^{-at})(1+\|u_t\|_{L^1(\underline{\Delta})})}, & t\in\underline{\Delta}_1:=[0,2], \quad (C)\\ u(t)=\zeta(t) & t\in\underline{\Delta}_2:=[-r,0]. \quad (D)\end{cases}$$
(FDPFDENVO2)

with

$$\Phi(t,u)=\frac{1}{4}t+\frac{3}{17(2+3u^4)}. \tag{39}$$

and

$$\Upsilon(t,u)=\frac{e^{at}u}{3(e^{at}+e^{-at})(1+u)}\quad (t,u)\in\underline{\Delta}_1\times(0,+\infty). \tag{40}$$

From equation (39) we can see that Φ is a CF on $\underline{\Delta}_1\times\mathbb{R}$ and $0<\Phi(t,y)<1$.

And from Eq.(40) we can see that

$$\begin{aligned}
&|\Upsilon(t,u)-\Upsilon(t,v)|\\
&\leq\frac{e^{at}}{3(e^{at}+e^{-at})}\left|\frac{\|u_t\|_{L^1(\underline{\Delta})}}{\left(1+\|u_t\|_{L^1(\underline{\Delta}_1)}\right)^2}-\frac{\|v_t\|_{L^1(\underline{\Delta}_1)}}{\left(1+\|v_t\|_{L^1(\underline{\Delta})}\right)^2}\right|\\
&\leq\frac{e^{at}}{3(e^{at}+e^{-at})}\frac{|\|u\|_{L^1(\underline{\Delta})}-\|v\|_{L^1(\underline{\Delta})}|}{\left(1+\|u_t\|_{L^1(\underline{\Delta})}\right)^2\left(1+\|v_t\|_{L^1(\underline{\Delta})}\right)^2}\\
&\leq\frac{e^{at}}{3(e^{at}+e^{-at})}\|u-v\|_{L^1(\underline{\Delta})}\\
&\leq\frac{1}{3}\|u-v\|_{L^1(\underline{\Delta})}.
\end{aligned} \tag{41}$$

It is easy to check that for the given choice of nonlinear functions, Φ and Υ assumptions (A1)-(A2) are satisfied. Therefore, by theorem (3.1), the problem (FDPFDENVO2) has a unique solution.

5 Conclusion

This paper investigates the Existence and Uniqueness of Solutions (EUSS) to a Finite Delay Problem for Fractional Differential Equations with Nonlinear Variable Order (FDPFDENVO). By employing Fixed Point Theory (FPT) (Theorems 3.1 and 3.2), we establish rigorous analytical results. Additionally, numerical simulations validate the theoretical findings. Our approach marks a substantial departure from conventional methods, providing a more efficient and computationally tractable framework. The proposed solution simplifies traditionally complex calculations, offering broad applicability across diverse

scientific domains, including engineering and applied mathematics.

6 Comments and Further work

We examined the existence of system (FDPFDENVO) using various theories and methodologies, and we provided an approval example to show that our criteria are practical. Generally, we complete our job with the stability of the solutions, but the note below will explain why this is not the case with us.

About the Choice of the Operators Our first comment revolves around the two options for perturbing the operators in order to obtain their existence and conclude the conditions. In the first proof, the perturbation was taken by a small enough parameter ε, but when it came to concluding the conditions, we suffered from taking aqequat choices. On the other hand, picking the second option in proof 2 makes everything flow smoothly and easily; hence, the second technique appears to be an accurate way to tackle problems involving (RLDFVO).

About the Stability The subject of the stability of such a situation remains unanswered, and it appears difficult to provide an answer to this important question.

Further Works In the future, we will address the existence and uniqueness of the infinite delay scenario, which is yet unknown, and we will only employ the second way because the first comment demonstrates that it is more powerful than the other.

Author Contributions These authors contributed equally to this work.

Funding This research has not received any funding.

Data Availibility Statement Our manuscript has no associated data.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

Agarwal P, El-Sayed AA, Tariboon J (2021) Vieta-Fibonacci operational matrices for spectral solutions of variable-order fractional integro-differential equations. J Comput Appl Math 382:113063

Agarwal P, Singh R, ul Rehman A (2021) Numerical solution of hybrid mathematical model of dengue transmission with relapse and memory via Adam-Bashforth-Moulton predictor-corrector scheme. Chaos, Solitons Fractals 143:110564

Ahmad I, Ali I, Jan R, Idris SA, Mousa M (2023) Solutions of a three-dimensional multi-term fractional anomalous solute transport model for contamination in groundwater. PLoS ONE 18(12):e0294348. https://doi.org/10.1371/journal.pone.0294348

Ait Mohammed H, Mezabia MEH, Tellab B, Amara AR, Emadifar, (2024) Analysis of Caputo fractional variable order multi-point initial value problems: existence, uniqueness, and stability. Bound Value Probl. https://doi.org/10.1186/s13661-024-01943-2

Almeida R, Tavares D, Torres DFM (2019) The variable-order fractional calculus of variations, 1st edn. Springer, Cham

Baleanu D, Diethelm K, Scalas E, Trujillo JJ (2012) Fractional calculus: models and numerical methods, 1st edn. World Scientific, Singapore

Benkerrouche A, Baleanu D, Souid MS, Hakem A, Inc M (2021) Boundary value problem for nonlinear fractional differential equations of variable order via Kuratowski MNC technique. Adv Differ Equ 2021:365

Benkerrouche A, Souid MS, Etemad S, Hakem A, Agarwal P, Rezapour S, Ntouyas SK, Tariboon J (2021) Qualitative study on solutions of a Hadamard variable order boundary problem via the Ulam-Hyers-Rassias stability. Fractal Fract 5:108

Benkerrouche A, Souid MS, Sitthithakerngkiet K, Hakem A (2021) Implicit nonlinear fractional differential equations of variable order. Bound Value Probl 2021:64

Benkerrouche A, Souid MS, Stamov G, Stamova I (2022) On the solutions of a quadratic integral equation of the Urysohn type of fractional variable order. Entropy 24:886

Benkerrouche A, Souid MS, Karapinar E, Hakem A (2022) On the boundary value problems of Hadamard fractional differential equations of variable order. Math Meth Appl Sci. https://doi.org/10.1002/mma.8306

Benkerrouche A, Souid MS, Stamov G, Stamova I (2022) Multiterm impulsive Caputo-Hadamard type differential equations of fractional variable order. Axioms 11:634

Bouazza Z, Souid MS, Hatira Günerhan H (2021) Multiterm boundary value problem of Caputo fractional differential equations of variable order. Adv Differ Equ 2021:1–17

Bouazza Z, Souid MS, Etemad S, Kaabar MKA (2022) Darbo fixed point criterion on solutions of a Hadamard nonlinear variable order problem and Ulam-Hyers-Rassias stability. J Funct Spaces 2022:1769359

Butt SI, Khan D, Jain S, Oros GI, Agarwal P, Momani S (2024) Fractional integral inequalities for Superquadratic functions via Atangana-Baleanu's operator with applications. Fractals (in press)

Garrappa R, Giusti A, Mainardi F (2021) Variable-order fractional calculus: a change of perspective. Commun Nonlinear Sci Numer Simul 102:105904

Hallouz A, Stamov G, Souid MS, Stamova I (2023) New results achieved for fractional differential equations with Riemann-Liouville derivatives of nonlinear variable order. Axioms 12(9):895

Haraux A, Cazenave T (1998) An introduction to semilinear evolution equations. Oxford Lecture Series in Mathematics and its Applications 13

Imtiaz A, Jan R, Normy Norfiza AR, Khan A, Abdeljawad T (2025) Numerical investigation of the dynamical behavior of Hepatitis B virus via Caputo-Fabrizio fractional derivative. Eur J Pure Appl Math 18(1):5509. https://doi.org/10.29020/nybg.ejpam.v18i1.5509

Imtiaz A, Jan R, Normy Norfiza AR, Khan A, Abdeljawad T (2025) Exploring fractional-order models in computational finance via an efficient hybrid approach. Eur J Pure Appl Math 18(1):5793. https://doi.org/10.29020/nybg.ejpam.v18i1.5793

Kesavan S (2004) Nonlinear functional analysis – a first course. Texts and readings in mathematics

Kesavan S (2019) Measure and Integration. In: Texts and readings in mathematics, vol 77, Springer

Kesavan S (2009) Functional analysis, 1st edn. Hindustan Book Agency, New Delhi
Kilbas AA, Srivastava HM, Trujillo JJ (2006) Theory and applications of fractional differential equations. North-Holland Mathematics Studies 204. Elsevier, Amsterdam
Kilbas A, Srivastava HM, Trujillo JJ (2006) Theory and applications of fractional differential equations, 1st edn, Elsevier, New York
Kıymaz IO, Agarwal P, Jain S, Çetinkaya A (2017) On a new extension of Caputo fractional derivative operator. Advances in real and complex analysis with applications. Springer, Singapore, pp 261–275
Lu X, Li H, Chen N (2019) An indicator for the electrode aging of lithium-ion batteries using a fractional variable order model. Electrochim Acta 299:378–387
Magin R (2006) Fractional calculus in bioengineering, 1st edn, Begell House, Redding
Odzijewicz T, Malinowska AB, Torres DFM (2013) Fractional variational calculus of variable order. In: Almeida A, Castro L, Speck FO (eds) Advances in harmonic analysis and operator theory. Operator theory: advances and applications, vol 229. Birkhäuser, Basel, pp 291–301
Patnaik S, Hollkamp JP, Semperlotti F (2020) Applications of variable-order fractional operators: a review. Proc R Soc A 476:20190498
Petráš I (2011) Fractional-order nonlinear systems, 1st edn, Springer, Heidelberg
Refice A, Souid MS, Stamova I (2021) On the boundary value problems of Hadamard fractional differential equations of variable order via Kuratowski MNC technique. Math 9:1134
Rekhviashvili S, Pskhu A, Agarwal P, Jain S (2019) Application of the fractional oscillator model to describe damped vibrations. Turk J Phys 43(3):236–242
Rezapour S, Souid MS, Etemad S, Bouazza Z, Ntouyas SK, Asawasamrit S, Tariboon J (2021) Mawhin continuation technique for a nonlinear BVP of variable order at resonance via piece-wise constant functions. Fractal Fract 5:216–230
Samko SG, Kilbas AA, Marichev OI (1993) Fractional integrals and derivatives: theory and applications, 1st edn, Gordon and Breach, Yverdon
Samko S (2013) Fractional integration and differentiation of variable order: an overview. Nonlinear Dyn 71:653–662
Samko S, Ross B (1993) Integration and differentiation to a variable fractional order. Integral Transform Spec Funct 1:277–300
Singh G, Agarwal P, Chand M, Jain S (2018) Certain fractional kinetic equations involving generalized k-Bessel function. Trans razmadze math inst 172(3):559–570
Stamova IM, Stamov GT (2017) Functional and impulsive differential equations of fractional order: qualitative analysis and applications, 1st edn, Taylor and Francis Group, Boca Raton
Sun HG, Chen W, Chen YQ (2009) Variable-order fractional differential operators in anomalous diffusion modeling. Phys A 388:4586–4592
Sweilam NH, AL-Mekhlafi SM, Alshomrani AS, Baleanu D (2020) Comparative study for optimal control nonlinear variable-order fractional tumor model. Chaos Solitons Fract 136:1–10
Tarasov VE (2015) Fractional dynamics: application of fractional calculus to dynamics of particles, fields and media, 1st edn, Springer, Beijing, China
Valerio D, Costa JS (2011) Variable-order fractional derivatives and their numerical approximations. Signal Process 91:470–483
Zhang S (2013) Existence of solutions for two-point boundary-value problems with singular differential equations of variable order. Electron J Differ Equ 2013:1–16
Zhang S, Hu L (2020) The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order. AIMS Math 5:2923–2943
Zhang S, Sun S, Hu L (2018) Approximate solutions to initial value problem for differential equation of variable order. J Fract Calc Appl 9:93–112
Zibar S, Tellab B, Amara A et al (2025) Existence, uniqueness and stability analysis of a nonlinear coupled system involving mixed ϕ-Riemann-Liouville and χ-Caputo fractional derivatives. Bound Value Probl. https://doi.org/10.1186/s13661-025-01994-z

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.