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ORIGINAL RESEARCH PAPER

Radii of starlikeness and convexity of extended generalized *k*-Bessel functions

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Abstract
The main purpose of this paper is to determine the radii of starlikeness and convexity of order β for three different normalizations of the extended generalized *k*-Bessel functions defined by

$$W^k_{v,a,d}(z) = \sum_{m\geq 0} \frac{(-d)^m}{\Gamma_k(mk+v+\frac{a+1}{2}k)\,\Gamma(m+1)} \left(\frac{z}{2}\right)^{2m+\frac{v}{k}},$$

where $k > 0$ and $v, a, d \in \mathbb{C}$. Moreover, we obtain tight lower and upper bounds for the radii of starlikeness and convexity of order zero. The characterization of entire functions in the Laguerre–Polya class plays a crucial role in this paper.

Keywords Extended generalized *k*-Bessel functions · Hadamard factorization · Convexity · Starlikeness · Euler Rayleigh inequalities · Laguerre–Polya class

Mathematics Subject Classification Primary 33C10 · Secondary 30C45

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1 Introduction and Preliminaries

In recent years, considerable attention has been given to the role of special functions especially in mathematics and physics due to their numerous applications. Because of the versatile properties of special functions, it is important to examine them from various perspectives. For recent studies on the geometric properties of Bessel functions, see [8, 9, 11, 13, 17–20, 24, 28–32, 35, 36].

Many mathematicians have recently focused on studying the properties of k-analogues of certain special functions (see [1–3, 6, 22, 23, 25–27, 40, 41]). Modified k-Bessel functions are closely related to classical modified Bessel functions, which are among the most important special functions.

In [23], the authors studied the k-gamma function and the Pochhammer k-symbol as generalizations of the classical gamma function and the classical Pochhammer symbol. Since then, many works have been devoted to generalizing various well-known special functions. Mondal and Akel [33] investigated Turán-type inequalities, monotonicity, and log-convexity properties, while Toklu [37] established geometric results for the generalized k-Bessel functions $W^k_{v,d}(z)$ with three parameters given by (1.5). More recently, Zayed [41] introduced and studied the extended generalized k-Bessel functions $W^k_{v,a,d}(z)$ with four parameters given by (1.8).

Let $\mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}$ denote the open disk of radius $r > 0$, and let $\mathbb{U} = \mathbb{U}_1$ be the open unit disk. Let $\mathcal{A}$ be the class of analytic functions f in $\mathbb{U}$ satisfying the usual normalization conditions $f(0) = 0$ and $f'(0) = 1$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$.

A function $f \in \mathcal{A}$ is said to be *starlike* in the disk $\mathbb{U}_r$ if it is univalent in $\mathbb{U}_r$ and the image $f(\mathbb{U}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin. Analytically, f is starlike in $\mathbb{U}_r$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}_r.$$

For $\beta \in [0, 1)$, the function f is said to be *starlike of order* β in $\mathbb{U}_r$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{U}_r.$$

The radius of starlikeness of order β for f is defined as the real number

$$r^*_\beta(f) = \sup\left\{r : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \text{ for all } z \in \mathbb{U}_r\right\}.$$

In particular, $r^*(f) = r^*_0(f)$ is the largest radius such that the image $f(\mathbb{U}_{r^*(f)})$ is a starlike domain with respect to the origin.

Similarly, a function $f \in \mathcal{A}$ is said to be *convex* in $\mathbb{U}_r$ if it is univalent in $\mathbb{U}_r$ and $f(\mathbb{U}_r)$ is a convex domain in $\mathbb{C}$. Analytically, f is convex in $\mathbb{U}_r$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\quad z\in\mathbb{U}_r.$$

For $\beta\in[0,1)$, f is said to be *convex of order* β in $\mathbb{U}_r$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta,\quad z\in\mathbb{U}_r.$$

The radius of convexity of order β for f is defined as the real number

$$r_\beta^c(f)=\sup\left\{r:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta \text{ for all } z\in\mathbb{U}_r\right\}.$$

In particular, $r^c(f)=r_0^c(f)$ is the largest radius such that $f(\mathbb{U}_{r^c(f)})$ is a convex domain.

1.1 The Laguerre–Pólya class of real entire functions

An entire function is a complex-valued function that is holomorphic on the entire complex plane. An entire function is called *real* if it maps the real axis into itself. A real entire function L belongs to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented in the form

$$L(x)=cx^me^{-bx^2+ax}\prod_{k\geq1}\left(1+\frac{x}{x_k}\right)e^{-\frac{x}{x_k}},$$

where $c,a,x_k\in\mathbb{R}$, $b\geq0$, $m\in\mathbb{N}\cup\{0\}$, and $\sum x_k^{-2}<\infty$. The class $\mathcal{LP}$ consists of real entire functions which are uniform limits on compact subsets of the complex plane of polynomials having only real zeros. It is important to note that $\mathcal{LP}$ is closed under differentiation; that is, if $L\in\mathcal{LP}$, then $L^{(n)}\in\mathcal{LP}$ for every $n\in\{1,2,3,\dots\}$.

1.2 Extended generalized k-Bessel functions

The Bessel function of the first kind of order v is defined by [34]

$$J_v(z)=\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+1)\Gamma(m+v+1)}\left(\frac{z}{2}\right)^{2m+v},\qquad z\in\mathbb{C}. \tag{1.1}$$

It is known that all the zeros of $J_v(z)$ are real for $v>-1$. The modified Bessel function of the first kind is defined by

$$I_v(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(m+1)\Gamma(m+v+1)}\left(\frac{z}{2}\right)^{2m+v},\qquad z\in\mathbb{C}, \tag{1.2}$$

and the spherical Bessel function of the first kind of order v is defined by

$$S_v(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+v+\frac{3}{2})\Gamma(m+1)} \left(\frac{z}{2}\right)^{2m+v}, \qquad z \in \mathbb{C}. \tag{1.3}$$

Baricz [7] investigated the second-order differential equation

$$z^2 w''(z) + azw'(z) + \left[(1-a)v + dz^2 - v^2\right] w(z) = 0 \tag{1.4}$$

for $v, a, d \in \mathbb{C}$, with the solution

$$W_{v,a,d}(z) = \sum_{m=0}^{\infty} \frac{(-d)^m}{\Gamma(m+v+\frac{a+1}{2})\Gamma(m+1)} \left(\frac{z}{2}\right)^{2m+v}, \qquad z \in \mathbb{C}.$$

The function $W_{v,a,d}$ generalizes the functions in Eqs. (1.1), (1.2) and (1.3). Another generalization was introduced by Mondal and Akel [33, p. 217], defined by the infinite series

$$W_{v,d}^k(z) = \sum_{m=0}^{\infty} \frac{(-d)^m}{\Gamma_k(mk+v+k)\Gamma(m+1)} \left(\frac{z}{2}\right)^{2m+\frac{v}{k}}, \qquad z \in \mathbb{C}, \tag{1.5}$$

where $k > 0$, $v > -1$, $d \in \mathbb{R}$ and Γ_k is the k-gamma function defined in [21, 23] by

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt, \qquad \Re(z) > 0,\ k > 0.$$

The k-gamma function and the classical gamma function are related via

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{1.6}$$

For $x \in \mathbb{C}$, $n \in \mathbb{N}^+$ and $k \in \mathbb{R}^+$, the k-gamma function satisfies the following identities:

$$\begin{aligned} &\Gamma_k(z+k) = z\Gamma_k(z), \qquad \Gamma_k(k) = 1, \\ &\frac{1}{\Gamma_k(z)} = zk^{-\frac{z}{k}} e^{\frac{z}{k}\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{z}{nk}\right) e^{-\frac{z}{nk}}, \\ &(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \end{aligned} \tag{1.7}$$

where γ is Euler's constant, and $(x)_{n,k}$ is the Pochhammer k-symbol defined by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k).$$

Note that $(x)_{n,1} = (x)_n$ is the classical Pochhammer symbol. In the special case $d = k = 1$, the function $W^k_{v,d}(z)$ (1.5) reduces to the classical Bessel function defined in (1.1).

More recently, Zayed [41] defined the extended generalized k-Bessel functions as

$$W^k_{v,a,d}(z) = \sum_{m=0}^{\infty} \frac{(-d)^m}{m!\Gamma_k(mk+v+\frac{a+1}{2}k)} \left(\frac{z}{2}\right)^{2m+\frac{v}{k}}, \qquad z \in \mathbb{C} \tag{1.8}$$

for $k > 0$ and $v, a, d \in \mathbb{C}$. Note that $W^1_{v,1,1}(z)$, $W^1_{v,1,-1}(z)$ and $W^k_{v,d}(z)$ reduce to the classical Bessel function, modified Bessel function and generalized k-Bessel function, respectively, as given in Eqs. (1.1), (1.2) and (1.5).

It is worth noting that $W^k_{v,a,d}(z)$ does not belong to the class $\mathcal{A}$. In this study, we define normalized forms of the extended generalized k-Bessel functions as follows:

$$\begin{aligned} f^k_{v,a,d}(z) &= \left(2^{\frac{v}{k}}\Gamma_k\left(v+\frac{a+1}{2}k\right) W^k_{v,a,d}(z)\right)^{\frac{k}{v}}, \\ g^k_{v,a,d}(z) &= 2^{\frac{v}{k}}\Gamma_k\left(v+\frac{a+1}{2}k\right) z^{1-\frac{v}{k}} W^k_{v,a,d}(z), \\ h^k_{v,a,d}(z) &= 2^{\frac{v}{k}}\Gamma_k\left(v+\frac{a+1}{2}k\right) z^{1-\frac{v}{2k}} W^k_{v,a,d}(\sqrt{z}). \end{aligned} \tag{1.9}$$

The main objective of this study is to determine the radii of starlikeness and convexity of order β for the functions $f^k_{v,a,d}$, $g^k_{v,a,d}$ and $h^k_{v,a,d}$. The following lemmas serve as key tools in proving our main results.

Lemma 1.1 *[41] Assume that $v, d, k > 0$ such that $\frac{v}{k} + \frac{a-1}{2} > 0$. Then the function $z \mapsto W^k_{v,a,d}(z)$ has infinitely many real zeros. Furthermore, its Weierstrass factorization is given by*

$$W^k_{v,a,d}(z) = \frac{\left(\frac{z}{2}\right)^{\frac{v}{k}}}{\Gamma_k\left(v+\frac{a+1}{2}k\right)} \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{({}_k\lambda_{\nu,a,d,n})^2}\right), \tag{1.10}$$

where ${}_k\lambda_{\nu,a,d,n}$ denotes the nth positive zero of $W^k_{v,a,d}(z)$. The above infinite product converges uniformly on compact subsets of $\mathbb{C}$.

Lemma 1.2 *Assume that $v, d, k > 0$ such that $\frac{v}{k} + \frac{a-1}{2} > 0$. Let ${}_k\lambda_{\nu,a,d,n}$ and ${}_k\lambda'_{\nu,a,d,n}$ denote the nth positive zeros of $W^k_{v,a,d}(z)$ and its derivative $\left(W^k_{v,a,d}(z)\right)'$, respectively. Then the positive zeros of $W^k_{v,a,d}(z)$ and its derivative interlace*

$${}_k\lambda'_{\nu,a,d,1} < {}_k\lambda_{\nu,a,d,1} < {}_k\lambda'_{\nu,a,d,2} < {}_k\lambda_{\nu,a,d,2} < {}_k\lambda'_{\nu,a,d,3} < {}_k\lambda_{\nu,a,d,3} < \cdots.$$

Proof Since $W^k_{v,a,d}(z)$ is an entire function (see [41]), its infinite product exists. By Hadamard theorem on the growth order of entire functions, the canonical factorization is exactly as stated in Lemma 1.1 . Computing the logarithmic derivative of the infinite product (1.10) gives

$$\frac{\left(W^k_{v,a,d}(z)\right)'}{W^k_{v,a,d}(z)} = \frac{v}{kz} - \sum_{m=1}^{\infty} \frac{2z}{({}_k\lambda_{\nu,a,d,m})^2 - z^2}.$$

Differentiating again, we find

$$\frac{d}{dz}\left[\frac{\left(W^k_{v,a,d}(z)\right)'}{W^k_{v,a,d}(z)}\right] = -\frac{v}{kz^2} - \sum_{m=1}^{\infty} \frac{2\left(({}_k\lambda_{\nu,a,d,m})^2 + z^2\right)}{\left(({}_k\lambda_{\nu,a,d,m})^2 - z^2\right)^2}.$$

Since the expression on the right-hand side is real and negative for $z \in \mathbb{R}$,

the quotient $\frac{\left(W^k_{v,a,d}(z)\right)'}{W^k_{v,a,d}(z)}$ is a strictly decreasing function from $+\infty$ to $-\infty$ as z increases through real values over the open interval $({}_k\lambda_{\nu,a,d,m}, {}_k\lambda_{\nu,a,d,m+1})$ for $m \in \mathbb{N}$. Hence, the function $z \longmapsto \left(W^k_{v,a,d}(z)\right)'$ vanishes just once between two consecutive zeros of the function $z \mapsto W^k_{v,a,d}(z)$. □

2 Main results

To the best of our knowledge, the first results concerning the starlikeness of Bessel functions were presented in [17, 31]. These two papers initiated the study of the univalence of Bessel functions and the determination of the radius of starlikeness for different types of normalizations. These findings raised further questions regarding other geometric properties of Bessel functions, such as convexity, uniform convexity, and more. Recently, for the functions $f^1_{v,1,1}(z)$, $g^1_{v,1,1}(z)$, and $h^1_{v,1,1}(z)$ in the case $v > -1$, Baricz et al. [8] obtained the radius of starlikeness of order β; Baricz and Szász [9] determined the radius of convexity of order β ; and Baricz et al. [13] established the radius of α- convexity of order β. Moreover, it is known that if $v \in (-2, -1)$, then the Bessel function has exactly two purely imaginary complex conjugate zeros, while all other zeros are real [39, p. 483]. For the functions $g^1_{v,1,1}(z)$ and $h^1_{v,1,1}(z)$ in the case $v \in (-2, -1)$, Baricz and Szász [9] and Szász [35] invesigated, respectively, the radius of convexity of order β and the radius of starlikeness of order β by using some inequalities. More recently, Baricz et al. [10] investigated the radii of starlikeness and convexity of order β for q-Bessel functions. Over the last two decades, this topic has been revisited, and numerous results have been estab-

lished for other special functions like Struve and Lommel [14], Wright [12], Mittag–Leffler [16], hyper-Bessel [5] and big q-Bessel functions [38] by using zeros.

In this section, we study the radii of starlikeness and convexity for the normalized extended generalized k-Bessel functions $f^k_{v,a,d}(z)$, $g^k_{v,a,d}(z)$ and $h^k_{v,a,d}(z)$. Very recently, Toklu [37] investigated the radii of starlikeness and convexity of order β for the normalized generalized k-Bessel functions $f^k_{v,1,d}(z)$, $g^k_{v,1,d}(z)$ and $h^k_{v,1,d}(z)$.

2.1 The radius of starlikeness of order β for the functions $f^k_{v,a,d}(z)$, $g^k_{v,a,d}(z)$ and $h^k_{v,a,d}(z)$

Motivated by the aforementioned research on geometric properties of special functions, we determine the radii of starlikeness of order β for the functions $f^k_{v,a,d}(z)$, $g^k_{v,a,d}(z)$ and $h^k_{v,a,d}(z)$. We also derive tight lower and upper bounds for these radii.

Theorem 2.1 *Let $k, d, v > 0$, $\frac{v}{k} + \frac{a-1}{2} > 0$ and $\beta \in [0, 1)$. The following statements hold*

(a) The radius of starlikeness of order β for the function $f^k_{v,a,d}$ is the smallest positive root of the equation

$$kr\left(W^k_{v,a,d}(r)\right)' - v\beta W^k_{v,a,d}(r) = 0.$$

(b) The radius of starlikeness of order β for the function $g^k_{v,a,d}$ is the smallest positive root of the equation

$$r\left(W^k_{v,a,d}(r)\right)' - \left(\beta + \frac{v}{k} - 1\right) W^k_{v,a,d}(r) = 0.$$

(c) The radius of starlikeness of order β for the function $h^k_{v,a,d}$ is the smallest positive root of the equation

$$\sqrt{r}\left(W^k_{v,a,d}(\sqrt{r})\right)' - \left(2\beta + \frac{v}{k} - 2\right) W^k_{v,a,d}(\sqrt{r}) = 0.$$

Proof We need to show that the following inequalities

$$\Re\left(\frac{z\left(f^k_{v,a,d}(z)\right)'}{f^k_{v,a,d}(z)}\right) > \beta, \quad \Re\left(\frac{z\left(g^k_{v,a,d}(z)\right)'}{g^k_{v,a,d}(z)}\right) > \beta \quad \text{and} \quad \Re\left(\frac{z\left(h^k_{v,a,d}(z)\right)'}{h^k_{v,a,d}(z)}\right) > \beta \tag{2.1}$$

hold for $z \in \mathbb{U}_{r^*_\beta}(f^k_{v,a,d})$, $z \in \mathbb{U}_{r^*_\beta}(g^k_{v,a,d})$ and $z \in \mathbb{U}_{r^*_\beta}(h^k_{v,a,d})$, respectively, and that each of the above inequalities does not hold in any larger disk.

It is also important to note that, under the corresponding conditions, all zeros of the function $W^k_{v,a,d}(z)$ are real.

Logarithmic differentiation of the functions defined in (1.9) yields

$$\frac{z\left(f^k_{v,a,d}(z)\right)'}{f^k_{v,a,d}(z)} = \frac{k}{\nu}\frac{z\left(W^k_{v,a,d}(z)\right)'}{W^k_{v,a,d}(z)} = 1 - \frac{2k}{\nu}\sum_{m=1}^{\infty}\frac{z^2}{\left({}_k\lambda_{\nu,a,d,m}\right)^2 - z^2},$$

$$\frac{z\left(g^k_{v,a,d}(z)\right)'}{g^k_{v,a,d}(z)} = 1 - \frac{\nu}{k} + \frac{z\left(W^k_{v,a,d}(z)\right)'}{W^k_{v,a,d}(z)} = 1 - 2\sum_{m=1}^{\infty}\frac{z^2}{\left({}_k\lambda_{\nu,a,d,m}\right)^2 - z^2},$$

$$\frac{z\left(h^k_{v,a,d}(z)\right)'}{h^k_{v,a,d}(z)} = 1 - \frac{\nu}{2k} + \frac{1}{2}\frac{\sqrt{z}\left(W^k_{v,a,d}(\sqrt{z})\right)'}{W^k_{v,a,d}(\sqrt{z})} = 1 - \sum_{m=1}^{\infty}\frac{z}{\left({}_k\lambda_{\nu,a,d,m}\right)^2 - z}.$$

From [9], it is known that if $z \in \mathbb{C}$ and $\mu \in \mathbb{R}$ such that $\mu > |z|$, then

$$\frac{|z|}{\mu - |z|} \geq \Re\left(\frac{z}{\mu - z}\right). \tag{2.2}$$

Therefore, the inequality

$$\frac{|z|^2}{{}_k\lambda^2_{\nu,a,d,m} - |z|^2} \geq \Re\left(\frac{z^2}{{}_k\lambda^2_{\nu,a,d,m} - z^2}\right)$$

holds for every $v > -1$, $m \in \mathbb{N}$ and $|z| < {}_k\lambda_{\nu,a,d,1}$. Hence,

$$\begin{aligned}
\Re\left(\frac{z\left(f_{v,a,d}^{k}(z)\right)'}{f_{v,a,d}^{k}(z)}\right) &= 1-\frac{2k}{\nu}\sum_{m=1}^{\infty}\Re\left(\frac{z^2}{\left({}_k\lambda_{\nu,a,d,m}\right)^2-z^2}\right)\\
&\geq 1-\frac{2k}{\nu}\sum_{m=1}^{\infty}\frac{|z|^2}{\left({}_k\lambda_{\nu,a,d,m}\right)^2-|z|^2}=\frac{|z|\left(f_{v,a,d}^{k}(|z|)\right)'}{f_{v,a,d}^{k}(|z|)},\\
\Re\left(\frac{z\left(g_{v,a,d}^{k}(z)\right)'}{g_{v,a,d}^{k}(z)}\right) &= 1-2\sum_{m=1}^{\infty}\Re\left(\frac{z^2}{\left({}_k\lambda_{\nu,a,d,m}\right)^2-z^2}\right)\\
&\geq 1-2\sum_{m=1}^{\infty}\frac{|z|^2}{\left({}_k\lambda_{\nu,a,d,m}\right)^2-|z|^2}=\frac{|z|\left(g_{v,a,d}^{k}(|z|)\right)'}{g_{v,a,d}^{k}(|z|)},\\
\Re\left(\frac{z\left(h_{v,a,d}^{k}(z)\right)'}{h_{v,a,d}^{k}(z)}\right) &= 1-\sum_{m=1}^{\infty}\Re\left(\frac{z}{\left({}_k\lambda_{\nu,a,d,m}\right)^2-z}\right)\\
&\geq 1-\sum_{m=1}^{\infty}\frac{|z|}{{}_k\lambda_{\nu,a,d,m}^2-|z|}=\frac{|z|\left(h_{v,a,d}^{k}(|z|)\right)'}{h_{v,a,d}^{k}(|z|)}.
\end{aligned}$$

These equalities are attained only when $z=|z|=r$. The last inequalities above, together with the minimum principle for harmonic functions, imply that the inequalities in (2.1) hold if and only if

$$|z|<{}_k x_{\nu,a,d,1},\quad |z|<{}_k y_{\nu,a,d,1}\quad\text{and}\quad |z|<{}_k z_{\nu,a,d,1},$$

respectively, where ${}_k x_{\nu,a,d,1}$, ${}_k y_{\nu,a,d,1}$ and ${}_k z_{\nu,a,d,1}$ are the smallest positive roots of the equations

$$\frac{r\left(f_{v,a,d}^{k}(r)\right)'}{f_{v,a,d}^{k}(r)}=\beta,\quad \frac{r\left(g_{v,a,d}^{k}(r)\right)'}{g_{v,a,d}^{k}(r)}=\beta,\quad\text{and}\quad \frac{r\left(h_{v,a,d}^{k}(r)\right)'}{h_{v,a,d}^{k}(r)}=\beta, \tag{2.3}$$

which are equivalent to

$$kr\left(W_{v,a,d}^{k}(r)\right)'-v\beta W_{v,a,d}^{k}(r)=0,\quad r\left(W_{v,a,d}^{k}(r)\right)'-\left(\beta+\frac{v}{k}-1\right)W_{v,a,d}^{k}(r)=0,$$

and

$$\sqrt{r}\left(W_{v,a,d}^{k}(\sqrt{r})\right)'-\left(2\beta+\frac{v}{k}-2\right)W_{v,a,d}^{k}(\sqrt{r})=0.$$

This completes the proof of the theorem. □

Remark 2.2 Taking $k = a = d = 1$ in Theorem 2.1, we recover the results in [9, Theorem 1]. If we take $a = 1$ in Theorem 2.1, we obtain the results given in [37].

The following theorem provides tight lower and upper bounds for the radii of starlikeness of the functions considered in Theorem 2.1. The main idea in the proof is to determine certain Euler Rayleigh inequalities for the first positive zero of some entire functions, which are connected to the transcendental equations appearing in Theorem 2.1.

Theorem 2.3 *Let $k, d, v > 0$ and $\frac{v}{k} + \frac{a-1}{2} > 0$. Then,*

(a) The radius of starlikeness $r^\star\left(f^k_{v,a,d}\right)$ satisfies the inequalities

$$\sqrt{\frac{2\left(2v+(a+1)k\right)}{dv(v+2k)}} < r^\star\left(f^k_{v,a,d}\right) < \sqrt{\frac{v^3(2k+v)\left(2v+(a+1)k\right)\left(2v+(a+3)k\right)}{dk\left(2(a+3)k^2+8kv+v^2\right)}},$$

$$\sqrt[4]{\frac{2v^2\left(2v+(a+1)k\right)\left(2v+(a+3)k\right)}{d^2k\left(2(3+a)k^2+8kv+v^2\right)}} < r^\star\left(f^k_{v,a,d}\right)$$

$$<2\sqrt{\frac{2kv\left(2v+(a+1)k\right)\left(2(a+3)k^2+8kv+v^2\right)}{d(v+2k)\left(8(a+3)k^3-4(a-7)k^2v-(a+5)kv^2-2v^3\right)}}.$$

(b) The radius of starlikeness $r^\star\left(g^k_{v,a,d}\right)$ satisfies the inequalities

$$2\sqrt{\frac{2v+(a+1)k}{6d}} < r^\star\left(g^k_{v,a,d}\right) < \sqrt{\frac{3\left(2v+(a+1)k\right)\left(2v+(a+3)k\right)}{d\left(4v+(11+2a)k\right)}},$$

$$2\sqrt[4]{\frac{\left(2v+(a+1)k\right)^2\left(2v+(a+3)k\right)}{8d^2\left(4v+(11+2a)k\right)}} < r^\star\left(g^k_{v,a,d}\right)$$

$$<2\sqrt{\frac{\left(2v+(a+1)k\right)^2\left(2v+(a+3)k\right)\left(2v+(a+5)k\right)}{d^2\left((-579+a(3+a(79+9a)))k^3+2(3+a(158+27a))k^2v+4(79+27a)kv^2+72v^3\right)}}.$$

(c) The radius of starlikeness $r^\star\left(h^k_{v,a,d}\right)$ satisfies the inequalities

$$\frac{2v+(a+1)k}{d} < r^\star\left(h^k_{v,a,d}\right) < \frac{4\left(2v+(a+1)k\right)\left(2v+(a+3)k\right)}{d\left(2v+(a+13)k\right)},$$

$$2\sqrt{\frac{\left(2v+(a+1)k\right)^2\left(2v+(a+3)k\right)}{d^2\left(2v+(a+13)k\right)}} < r^\star\left(h^k_{v,a,d}\right)$$

$$<\frac{2\left(2v+(a+13)k\right)\left(2v+(a+1)k\right)^2\left(2v+(a+3)k\right)\left(2v+(a+5)k\right)}{d^2\left((-597+a(-39+a(49+3a)))k^3+2(-39+a(98+9a))k^2v+4(49+9a)kv^2+24v^3\right)}.$$

Proof *(a)* The infinite series representation of the function $\Theta^k_{v,a,d}(z) = \left(W^k_{v,a,d}(z)\right)'$ and its derivative are given by

$$\Theta^k_{v,a,d}(z) = \left(W^k_{v,a,d}(z)\right)' = \sum_{m=0}^{\infty} \frac{(-d)^m \left(2m + \frac{v}{k}\right)}{2^{2m+\frac{v}{k}} m! \Gamma_k(mk + v + \frac{a+1}{2}k)} z^{2m+\frac{v}{k}-1} \tag{2.4}$$

and

$$\left(\Theta^k_{v,a,d}(z)\right)' = \sum_{m=0}^{\infty} \frac{(-d)^m \left(2m + \frac{v}{k}\right)\left(2m + \frac{v}{k} - 1\right)}{2^{2m+\frac{v}{k}} m! \Gamma_k(mk + v + \frac{a+1}{2}k)} z^{2m+\frac{v}{k}-2}. \tag{2.5}$$

By Lemma 1.1, the function $z \mapsto z^{1-\frac{v}{k}} \Theta^k_{v,a,d}(z)$ belongs to the Laguerre-Polya class $\mathcal{LP}$. Therefore, all zeros of the function $\Theta^k_{v,a,d}$ are real. Let ${}_k\lambda'_{\nu,a,d,m}$ denote the positive zeros of $\Theta^k_{v,a,d}$.

The function $\Theta^k_{v,a,d}(z)$ can be expressed as

$$\Theta^k_{v,a,d}(z) = \frac{v \left(\frac{z}{2}\right)^{\frac{v}{k}-1}}{2k\Gamma_k\left(v + \frac{a+1}{2}k\right)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{\left({}_k\lambda'_{\nu,a,d,m}\right)^2}\right). \tag{2.6}$$

Taking the logarithmic derivative of (2.6) yields

$$\begin{aligned} \frac{z\left(\Theta^k_{v,a,d}(z)\right)'}{\Theta^k_{v,a,d}(z)} - \left(\frac{v}{k} - 1\right) &= -2\sum_{m=1}^{\infty} \frac{z^2}{\left({}_k\lambda'_{\nu,a,d,m}\right)^2 - z^2} \\ &= -2\sum_{m=1}^{\infty}\sum_{t=0}^{\infty} \frac{z^{2t+2}}{\left({}_k\lambda'_{\nu,a,d,m}\right)^{2t+2}} \\ &= -2\sum_{t=0}^{\infty} \delta_{t+1} z^{2t+2}, \end{aligned} \tag{2.7}$$

where $\delta_t = \sum_{m=1}^{\infty} \left({}_k\lambda'_{\nu,a,d,m}\right)^{-2t}$.

On the other hand, define

$$a_m = \frac{(-d)^m \left(2m + \frac{v}{k}\right)\left(2m + \frac{v}{k} - 1\right)}{2^{2m} m! \Gamma_k(mk + v + \frac{a+1}{2}k)}, \quad b_m = \frac{(-d)^m \left(2m + \frac{v}{k}\right)}{2^{2m} m! \Gamma_k(mk + v + \frac{a+1}{2}k)}.$$

Using Eqs. (2.4) and (2.5), we get

$$\frac{z\left(\Theta_{v,a,d}^{k}(z)\right)'}{\Theta_{v,a,d}^{k}(z)}=\frac{\sum_{m=0}^{\infty}a_m z^{2m}}{\sum_{m=0}^{\infty}b_m z^{2m}}. \tag{2.8}$$

Comparing the coefficients of (2.7) and (2.8), we obtain

$$\begin{aligned}
\delta_1 &= \frac{dv(v+2k)}{2\left(2v+(a+1)k\right)},\\
\delta_2 &= \frac{d^2k\left(2(3+a)k^2+8kv+v^2\right)}{2v^2\left(2v+(a+1)k\right)^2\left(2v+(a+3)k\right)},\\
\delta_3 &= \frac{-d^3(v+2k)\left(-8(a+3)k^3+4(-7+a)k^2v+(5+a)kv^2+2v^3\right)}{16v^3\left(2v+(a+1)k\right)^3\left(2v+(a+3)k\right)}.
\end{aligned}$$

By the Euler–Rayleigh inequalities

$$\delta_t^{-\frac{1}{t}}<\left({}_k\lambda'_{\nu,a,d,1}\right)^2<\frac{\delta_t}{\delta_{t+1}}$$

for $t=1,2$, which completes the proof of part *(a)*.

(b) For $\beta=0$, from the second part of Theorem 2.1, we know that the radius of starlikeness of order zero is the smallest positive root of the equation $\left(g_{v,a,d}^{k}(z)\right)'=0$.

Therefore, we study the first positive zero of the function

$$\Psi_{v,a,d}^{k}(z)=\frac{dg_{v,a,d}^{k}}{dz}(2\sqrt{z})=\Gamma_k\left(v+\frac{a+1}{2}k\right)\sum_{m=0}^{\infty}\frac{(-d)^m(2m+1)}{m!\Gamma_k(mk+v+\frac{a+1}{2}k)}z^m.$$

Since $W_{v,a,d}^{k}$ belongs to the class $\mathcal{LP}$ and $\mathcal{LP}$ is closed under differentiation, it follows that $\Psi_{v,a,d}^{k}$ also belongs to $\mathcal{LP}$. Thus, all zeros of $\Psi_{v,a,d}^{k}$ are real.

Let ${}_kw_{\nu,a,d,m}$ denote the positive zeros of $\Psi_{v,a,d}^{k}$. Then, the function $\Psi_{v,a,d}^{k}$ admits the infinite product representation

$$\Psi_{v,a,d}^{k}(z)=\prod_{m=1}^{\infty}\left(1-\frac{z}{{}_kw_{\nu,a,d,m}}\right). \tag{2.9}$$

Taking the logarithmic derivative of both sides of (2.9), we obtain

$$\begin{aligned}
\frac{\left(\Psi_{v,a,d}^{k}(z)\right)'}{\Psi_{v,a,d}^{k}(z)} &= -\sum_{m=1}^{\infty}\frac{1}{{}_kw_{\nu,a,d,m}-z}=-\sum_{m=1}^{\infty}\sum_{t=0}^{\infty}\frac{z^t}{\left({}_kw_{\nu,a,d,m}\right)^{t+1}}\\
&= -\sum_{t=0}^{\infty}l_{t+1}z^t
\end{aligned} \tag{2.10}$$

for

$|z| < {}_k w_{\nu,a,d,1}$, where $l_t = \sum_{m=1}^{\infty} \left({}_k w_{\nu,a,d,m}\right)^{-t}$.

Also, considering the fact that

$$\left(\Psi^k_{v,a,d}(z)\right)' = \Gamma_k\left(v + \frac{a+1}{2}k\right) \sum_{m=0}^{\infty} \frac{(-d)^{m+1}(2m+3)}{m!\,\Gamma_k((m+1)k + v + \frac{a+1}{2}k)} z^m,$$

we obtain

$$\frac{\left(\Psi^k_{v,a,d}(z)\right)'}{\Psi^k_{v,a,d}(z)} = \frac{\sum_{m=0}^{\infty} c_m z^m}{\sum_{m=0}^{\infty} d_m z^m}, \tag{2.11}$$

where

$$c_m = \frac{(-d)^{m+1}(2m+3)}{m!\,\Gamma_k((m+1)k + v + \frac{a+1}{2}k)}$$

and

$$d_m = \frac{(-d)^m(2m+1)}{m!\,\Gamma_k(mk + v + \frac{a+1}{2}k)}.$$

By comparing the coefficients of equations (2.10) and (2.11), we obtain

$$l_1 = \frac{6d}{2v + (a+1)k},$$

$$l_2 = \frac{8d^2\,(4v + (11+2a)k)}{(2v + (a+1)k)^2\,(2v + (a+3)k)},$$

$$l_3 = \frac{8d^4\left((-579 + a(3 + a(79+9a)))k^3 + 2(3 + a(158+27a))k^2 v + 4(79+27a)kv^2 + 72v^3\right)}{(2v + (a+1)k)^4\,(2v + (a+3)k)^2\,(2v + (a+5)k)}.$$

Using the Euler–Rayleigh inequalities

$$l_t^{-1/t} < {}_k w_{\nu,a,d,1} < \frac{l_t}{l_{t+1}}$$

for

$t = 1, 2$, we obtain the bounds for the radius of starlikeness given in part *(b)*.

(c) Consider the entire function

$$\Omega^k_{v,a,d}(z) = \frac{dh^k_{v,a,d}}{dz}(4z) = \Gamma_k\left(v + \frac{a+1}{2}k\right) \sum_{m=0}^{\infty} \frac{(-d)^m(m+1)}{m!\Gamma_k(km + v + \frac{a+1}{2}k)} z^m. \tag{2.12}$$

By Lemma 1.2, and since the Laguerre Polya class $\mathcal{LP}$ is closed under differentiation, the function $\Omega_{v,a,d}^{k}$ also belongs to $\mathcal{LP}$. Therefore, we may write

$$\Omega_{v,a,d}^{k}(z)=\prod_{m=1}^{\infty}\left(1-\frac{z}{{}_{k}\zeta_{\nu,a,d,m}}\right), \tag{2.13}$$

where ${}_{k}\zeta_{\nu,a,d,m}$ denote the positive zeros of $\Omega_{v,a,d}^{k}(z)$.

Taking the logarithmic derivative of (2.13), we obtain

$$\begin{aligned}\frac{\left(\Omega_{v,a,d}^{k}(z)\right)'}{\Omega_{v,a,d}^{k}(z)}&=-\sum_{m=1}^{\infty}\frac{1}{{}_{k}\zeta_{\nu,a,d,m}-z}=-\sum_{m=1}^{\infty}\sum_{t=0}^{\infty}\frac{z^{t}}{\left({}_{k}\zeta_{\nu,a,d,m}\right)^{t+1}}\\&=-\sum_{t=0}^{\infty}\kappa_{t+1}z^{t},\quad |z|<{}_{k}\zeta_{\nu,a,d,1},\end{aligned} \tag{2.14}$$

where

$$\kappa_{t}=\sum_{m=1}^{\infty}\left({}_{k}\zeta_{\nu,a,d,m}\right)^{-t}.$$

On the other hand, we have

$$\frac{\left(\Omega_{v,a,d}^{k}(z)\right)'}{\Omega_{v,a,d}^{k}(z)}=\frac{\sum_{m=0}^{\infty}\frac{(-d)^{m+1}(m+2)}{m!\,\Gamma_{k}(k(m+1)+v+\frac{a+1}{2}k)}z^{m}}{\sum_{m=0}^{\infty}\frac{(-d)^{m}(m+1)}{m!\,\Gamma_{k}(km+v+\frac{a+1}{2}k)}z^{m}}. \tag{2.15}$$

Using the Eqs. (2.14) and (2.15), we can express the Euler–Rayleigh sums in terms of the parameters v, a, d, and k. By applying the Euler–Rayleigh inequalities

$$\left(\kappa_{t}\right)^{-1/t}<{}_{k}\zeta_{\nu,a,d,1}<\frac{\kappa_{t}}{\kappa_{t+1}},$$

we obtain the following bounds

$$4\left(\kappa_{t}\right)^{-1/t}<r^{\star}\left(h_{v,a,d}^{k}\right)<4\frac{\kappa_{t}}{\kappa_{t+1}}$$

for

${}_{4k}\zeta_{\nu,a,d,1}$ and $t\in\{1,2\}$. In particular, we have

$$\kappa_1 = \frac{4d}{2v + (a+1)k},$$
$$\kappa_2 = \frac{4d^2\,(2v + (a+13)k)}{(2v + (a+1)k)^2(2v + (a+3)k)},$$

and

$$\kappa_3 = \frac{8d^4\left((-597 + a(-39 + a(49 + 3a)))k^3 + 2(-39 + a(98 + 9a))k^2v + 4(49 + 9a)kv^2 + 24v^3\right)}{(2v + (a+1)k)^4(2v + (a+3)k)^2(2v + (a+5)k)}.$$

Thus, the inequalities given in part *(c)* are obtained. □

Remark 2.4 It is obvious that the main results presented in Theorem 2.3 , when taking $k = a = d = 1$, coincide with the inequalities given in [4] . If we choose $a = 1$ in Theorem 2.3, we obtain the results presented in [37].

2.2 The radius of convexity of order β of the functions $f^{k}_{v,a,d}(z)$, $g^{k}_{v,a,d}(z)$ and $h^{k}_{v,a,d}(z)$

Theorem 2.5 *Let $k > 0$, $d > 0$, $v > 0$, $\frac{v}{k} + \frac{a-1}{2} > 0$ and $\beta \in [0, 1)$. Then,*

(a) The radius of convexity of order β of the function $f^{k}_{v,a,d}$ is the smallest positive root of the equation

$$1 + r\frac{\left(W^{k}_{v,a,d}(r)\right)''}{\left(W^{k}_{v,a,d}(r)\right)'} + \left(\frac{k}{v} - 1\right) r\frac{\left(W^{k}_{v,a,d}(r)\right)'}{W^{k}_{v,a,d}(r)} = \beta.$$

(b) The The radius of convexity of order β of the function $g^{k}_{v,a,d}$ is the smallest positive root of the equation

$$1 + r\frac{\left(g^{k}_{v,a,d}(r)\right)''}{\left(g^{k}_{v,a,d}(r)\right)'} = \beta.$$

(c) The radius of convexity of order β of the function $h^{k}_{v,a,d}$ is the smallest positive root of the equation

$$1 + r\frac{\left(h^{k}_{v,a,d}(r)\right)''}{\left(h^{k}_{v,a,d}(r)\right)'} = \beta.$$

Proof *(a)* Observe that

$$1+z\frac{\left(f_{v,a,d}^{k}(z)\right)''}{\left(f_{v,a,d}^{k}(z)\right)'}=1+z\frac{\left(W_{v,a,d}^{k}(z)\right)''}{\left(W_{v,a,d}^{k}(z)\right)'}+\left(\frac{k}{v}-1\right)z\frac{\left(W_{v,a,d}^{k}(z)\right)'}{W_{v,a,d}^{k}(z)}.$$

By means of Eqs. (1.10) and (2.6), we obtain

$$1+z\frac{\left(f_{v,a,d}^{k}(z)\right)''}{\left(f_{v,a,d}^{k}(z)\right)'}=1-2\left(\frac{k}{v}-1\right)\sum_{m=1}^{\infty}\frac{z^2}{\left({}_k\lambda_{\nu,a,d,m}\right)^2-z^2}-2\sum_{m=1}^{\infty}\frac{z^2}{\left({}_k\lambda'_{\nu,a,d,m}\right)^2-z^2}.$$

Assume now that $k>v$. Using inequality (2.2), for all $z\in\mathbb{U}_{{}_k\lambda'_{\nu,a,d,1}}$, we obtain

$$\Re\left(1+z\frac{\left(f_{v,a,d}^{k}(z)\right)''}{\left(f_{v,a,d}^{k}(z)\right)'}\right)\geq 1-\left(\frac{k}{v}-1\right)\sum_{m=1}^{\infty}\frac{2r^2}{\left({}_k\lambda_{\nu,a,d,m}\right)^2-r^2}-\sum_{m=1}^{\infty}\frac{2r^2}{\left({}_k\lambda'_{\nu,a,d,m}\right)^2-r^2},\tag{2.16}$$

where $|z|=r$.

Moreover, using the inequality (see [9, Lemma 2.1])

$$\alpha\Re\left(\frac{z}{a-z}\right)-\Re\left(\frac{z}{b-z}\right)\geq\alpha\frac{|z|}{a-|z|}-\frac{|z|}{b-|z|},\quad \text{for } a>b>0,\ \alpha\in[0,1],\ |z|<b,$$

we conclude that inequality (2.16) also holds when $k<v$ for all $z\in\mathbb{U}_{{}_k\lambda'_{\nu,a,d,1}}$.Thus, for $r\in(0,{}_k\lambda'_{\nu,a,d,1})$, we have

$$\inf_{z\in\mathbb{U}_r}\left\{\Re\left(1+z\frac{\left(f_{v,a,d}^{k}(z)\right)''}{\left(f_{v,a,d}^{k}(z)\right)'}\right)\right\}=1+r\frac{\left(f_{v,a,d}^{k}(r)\right)''}{\left(f_{v,a,d}^{k}(r)\right)'}.$$

Now define the function $F_{\nu,a,d}^{k}:(0,{}_k\lambda'_{\nu,a,d,1})\to\mathbb{R}$ as

$$F_{\nu,a,d}^{k}(r)=1+r\frac{\left(f_{v,a,d}^{k}(r)\right)''}{\left(f_{v,a,d}^{k}(r)\right)'}.$$

The function is strictly decreasing since

$$\begin{aligned}\left(F^k_{\nu,a,d}(r)\right)' =& -4r\left(\frac{k}{v}-1\right)\sum_{m=1}^{\infty}\frac{\left({}_k\lambda_{\nu,a,d,m}\right)^2}{\left(\left({}_k\lambda_{\nu,a,d,m}\right)^2-r^2\right)^2}\\ &-4r\sum_{m=1}^{\infty}\frac{\left({}_k\lambda'_{\nu,a,d,m}\right)^2}{\left(\left({}_k\lambda'_{\nu,a,d,m}\right)^2-r^2\right)^2}\\ <&4r\sum_{m=1}^{\infty}\frac{\left({}_k\lambda_{\nu,a,d,m}\right)^2}{\left(\left({}_k\lambda_{\nu,a,d,m}\right)^2-r^2\right)^2}\\ &-4r\sum_{n=1}^{\infty}\frac{\left({}_k\lambda'_{\nu,a,d,m}\right)^2}{\left(\left({}_k\lambda'_{\nu,a,d,m}\right)^2-r^2\right)^2}<0\end{aligned}$$

for $r\in\left(0,\ {}_k\lambda'_{\nu,a,d,m}\right)$. Here, we have again used the interlacing property of the zeros stated in Lemma 1.2. Also, taking into consideration that

$$\lim_{r\searrow 0}F^k_{\nu,a,d}(r)=1\quad\text{and}\quad\lim_{r\nearrow {}_k\lambda_{\nu,a,d,1}}F^k_{\nu,a,d}(r)=-\infty,$$

we conclude that for $z\in\mathbb{U}_{r_1}$, the following inequality holds

$$\Re\left(1+z\frac{\left(f^k_{v,a,d}(z)\right)''}{\left(f^k_{v,a,d}(z)\right)'}\right)>\beta,$$

if and only if r_1 is the unique root of

$$1+r\frac{\left(f^k_{v,a,d}(r)\right)''}{\left(f^k_{v,a,d}(r)\right)'}=\beta,$$

located in the interval $\left(0,\ {}_k\lambda'_{\nu,a,d,1}\right)$.

(b) Since $g^k_{v,a,d}\in\mathcal{LP}$, it follows that $\left(g^k_{v,a,d}\right)'\in\mathcal{LP}$. Moreover, since both have order of growth equal to $\frac{1}{2}$ (see [41]), the Hadamard factorization theorem implies the Weierstrassian canonical representation

$$\left(g^k_{v,a,d}(z)\right)'=\prod_{m=1}^{\infty}\left(1-\frac{z^2}{\left({}_k\sigma_{\nu,a,d,m}\right)^2}\right),$$

where ${}_k\sigma_{\nu,a,d,m}$ denotes the mth positive zero of the function $\left(g^k_{v,a,d}(z)\right)'$.

Thus, we have

$$1+z\frac{\left(g^k_{v,a,d}(z)\right)''}{\left(g^k_{v,a,d}(z)\right)'}=1-2\sum_{m=1}^{\infty}\frac{z^2}{\left({}_k\sigma_{\nu,a,d,m}\right)^2-z^2}.$$

Applying inequality (2.2), we obtain for $|z|=r$

$$\Re\left(1+z\frac{\left(g^k_{v,a,d}(z)\right)''}{\left(g^k_{v,a,d}(z)\right)'}\right)\geq 1-2\sum_{m=1}^{\infty}\frac{r^2}{\left({}_k\sigma_{\nu,a,d,m}\right)^2-r^2}.$$

Therefore, for $r\in(0,\ {}_k\sigma_{\nu,a,d,1})$, we obtain

$$\inf_{z\in\mathbb{U}_r}\Re\left(1+z\frac{\left(g^k_{v,a,d}(z)\right)''}{\left(g^k_{v,a,d}(z)\right)'}\right)=1-2\sum_{m=1}^{\infty}\frac{r^2}{\left({}_k\sigma_{\nu,a,d,m}\right)^2-r^2}=1+r\frac{\left(g^k_{v,a,d}(r)\right)''}{\left(g^k_{v,a,d}(r)\right)'}.$$

Consider the function

$$G^k_{\nu,a,d}:(0,\ {}_k\sigma_{\nu,a,d,1})\to\mathbb{R},$$

defined by

$$G^k_{\nu,a,d}(r)=1-\beta+r\frac{\left(g^k_{v,a,d}(r)\right)''}{\left(g^k_{v,a,d}(r)\right)'}.$$

This function is strictly decreasing and satisfies the limits

$$\lim_{r\nearrow {}_k\sigma_{\nu,a,d,1}}G^k_{\nu,a,d}(r)=-\infty\quad\text{and}\quad\lim_{r\searrow 0}G^k_{\nu,a,d}(r)=1-\beta,$$

which implies that for $z\in\mathbb{U}_{r_2}$, the inequality

$$\Re\left(1+z\frac{\left(g^k_{v,a,d}(z)\right)''}{\left(g^k_{v,a,d}(z)\right)'}\right)>\beta$$

holds if and only if r_2 is the unique root of the equation

$$1+r\frac{\left(g^k_{v,a,d}(r)\right)''}{\left(g^k_{v,a,d}(r)\right)'}=\beta,$$

located in the interval $(0,\ {}_k\sigma_{\nu,a,d,1})$.

(c) We know that the functions $h^k_{v,a,d}(z)$ and $\left(h^k_{v,a,d}(z)\right)'$ belong to the class $\mathcal{LP}$. Moreover, in light of Laguerre's lemma stated in [15], it follows that $\left(h^k_{v,a,d}(z)\right)'$ has only positive real zeros. Also, since both have order of growth equal to $\frac{1}{2}$ (see [41]), we have the following infinite product representation

$$\left(h^k_{v,a,d}(z)\right)'=\prod_{m=1}^{\infty}\left(1-\frac{z}{{}_k\varrho_{\nu,a,d,m}}\right),$$

where ${}_k\varrho_{\nu,a,d,m}$ denotes the mth positive zero of the function $\left(h^k_{v,a,d}(z)\right)'$.

Thus, we have

$$1+z\frac{\left(h^k_{v,a,d}(z)\right)''}{\left(h^k_{v,a,d}(z)\right)'}=1-\sum_{m=1}^{\infty}\frac{z}{{}_k\varrho_{\nu,a,d,m}-z}.$$

Applying inequality (2.2), we obtain for $|z|=r$

$$\Re\left(1+z\frac{\left(h^k_{v,a,d}(z)\right)''}{\left(h^k_{v,a,d}(z)\right)'}\right)\geq 1-\sum_{m=1}^{\infty}\frac{r}{{}_k\varrho_{\nu,a,d,m}-r}.$$

Therefore, for $r\in(0,\ {}_k\varrho_{\nu,a,d,1})$, we obtain

$$\inf_{z\in\mathbb{U}_r}\Re\left(1+z\frac{\left(h^k_{v,a,d}(z)\right)''}{\left(h^k_{v,a,d}(z)\right)'}\right)=1+r\frac{\left(h^k_{v,a,d}(r)\right)''}{\left(h^k_{v,a,d}(r)\right)'}.$$

Consider the function

$$P^k_{\nu,a,d}:(0,\ {}_k\varrho_{\nu,a,d,1})\to\mathbb{R},$$

defined by

$$P^k_{\nu,a,d}(r) = 1 - \beta + r\frac{\left(h^k_{v,a,d}(r)\right)''}{\left(h^k_{v,a,d}(r)\right)'}.$$

This function is strictly decreasing and satisfies the limits

$$\lim_{r \nearrow {}_k\varrho_{\nu,a,d,1}} P^k_{\nu,a,d}(r) = -\infty \quad \text{and} \quad \lim_{r \searrow 0} P^k_{\nu,a,d}(r) = 1 - \beta,$$

which implies that for $z \in \mathbb{U}_{r_3}$, the inequality

$$\Re\left(1 + r\frac{\left(h^k_{v,a,d}(r)\right)''}{\left(h^k_{v,a,d}(r)\right)'}\right) > \beta,$$

holds if and only if r_3 is the unique root of the equation

$$1 + r\frac{\left(h^k_{v,a,d}(r)\right)''}{\left(h^k_{v,a,d}(r)\right)'} = \beta,$$

located in the interval $(0,_k \varrho_{\nu,a,d,1})$. Thus, the proof of the theorem is complete. □

Remark 2.6 Taking $k = a = d = 1$ in Theorem 2.5, the results correspond to those given in [9, Theorems 1, 2, and 3]. If we choose $a = 1$ in Theorem 2.5, the results correspond to those presented in [37].

Theorem 2.7 *Let $k > 0$, $d > 0$, $v > 0$ and $\frac{v}{k} + \frac{a-1}{2} > 0$. Then,*

(a) The radius of convexity $r^c(g^k_{v,a,d})$ satisfies the inequalities

$$\frac{2}{3}\sqrt{\frac{2v + (a+1)k}{2d}} < r^c(g^k_{v,a,d}) < 3\sqrt{\frac{(2v + (a+1)k)(2v + (a+3)k)}{d(56v + (109 + 28a)k)}},$$

$$2\sqrt[4]{\frac{(2v + (a+1)k)^2(2v + (a+3)k)}{8d^2(56v + (109 + 28a)k)}} < r^c(g^k_{v,a,d})$$

$$< 2\sqrt{\frac{(56v + (109 + 28a)k)(2v + (a+1)k)^2(2v + (a+3)k)(2v + (a+5)k)}{2d^2\Big((-236719 - a(167710 + a(39011 + 3044a)))k^3 - 4(83855 + a(39011 + 4566a))k^2v - 4(39011 + 9132a)k^2v - 24352v^3\Big)}}.$$

(b) The radius of convexity $r^c(h^k_{v,a,d})$ satisfies the inequalities

$$\frac{2v+(a+1)k}{2d} < r^c(h^k_{v,a,d}) < \frac{(2v+(a+1)k)(2v+(a+3)k)}{4d(16v+(-1+8a)k)},$$

$$\frac{(2v+(a+1)k)^2(2v+(a+3)k)}{8d^2(16v+(1-8a)k)} < r^c(h^k_{v,a,d})$$

$$< \frac{256\,(2v+(a+1)k)^2\,(2v+(a+3)k)\,(2v+(a+5)k)\,(16v+(1-8a)\,k)}{d^2\,((14997+a(8059+3a(445+27a)))\,k^3+2(8059+3a(890+81a))k^2v+12(445+81a)kv^2+648v^3)}.$$

Proof *(a)* By applying the Alexander duality theorem for starlike and convex functions, we conclude that the function $g^k_{v,a,d}(z)$ is convex if and only if $z\left(g^k_{v,a,d}(z)\right)'$ is starlike. According to Theorems 2.1 and 2.5, the smallest positive zero of the function $z \mapsto \left(z\left(g^k_{v,a,d}(z)\right)'\right)'$ is, in fact, the radius of starlikeness of $z \mapsto z\left(g^k_{v,a,d}(z)\right)'$. Therefore, the radius of convexity $r^c\left(g^k_{v,a,d}\right)$ is the smallest positive root of the equation $\left(z\left(g^k_{v,a,d}(z)\right)'\right)' = 0$.

Now, using (1.10), we have

$$\left(z\left(g^k_{v,a,d}(z)\right)'\right)' = \Gamma_k(v+\frac{a+1}{2}k)\sum_{m=0}^{\infty}\frac{(-d)^m\,(2m+1)^2}{2^{2m}m!\Gamma_k(mk+v+\frac{a+1}{2}k)}z^{2m}.$$

Furthermore, substituting $2\sqrt{z}$ in place of z yields

$$\Xi^k_{v,a,d}(z) = \Gamma_k(v+\frac{a+1}{2}k)\sum_{m=0}^{\infty}\frac{(-d)^m\,(2m+1)^2}{m!\Gamma_k(mk+v+\frac{a+1}{2}k)}z^{m}. \tag{2.17}$$

Since the function $g^k_{v,a,d}(z)$ belongs to the Laguerre–Pólya class of entire functions, and since $\mathcal{LP}$ is closed under differentiation, we have $\Xi^k_{v,a,d}(z) \in \mathcal{LP}$. Thus, $\Xi^k_{v,a,d}(z)$ is an entire function with only real zeros.

Moreover, by Laguerre's theorem (see [15]), the function $\Xi^k_{v,a,d}(z)$ has real and positive zeros. It also admits the following infinite product representation

$$\Xi^k_{v,a,d}(z) = \prod_{m=1}^{\infty}\left(1-\frac{z}{{}_k\varkappa_{\nu,a,d,m}}\right), \quad (|z| <_k {}_k\varkappa_{\nu,a,d,1})$$

where ${}_k\varkappa_{\nu,a,d,m}$ is the mth positive zero of $\Xi^k_{v,a,d}(z)$. Consequently, we have

$$\frac{\left(\Xi^k_{v,a,d}(z)\right)'}{\Xi^k_{v,a,d}(z)} = -\sum_{m=1}^{\infty}\frac{1}{{}_k\varkappa_{\nu,a,d,m}-z} = -\sum_{s=0}^{\infty}\varpi_{s+1}z^s, \quad (|z| <_k {}_k\varkappa_{\nu,a,d,1}), \tag{2.18}$$

where $\varpi_s = \sum_{m=1}^{\infty}\left({}_k\varkappa_{\nu,a,d,m}\right)^{-s}$.

On the other hand, from (2.17), we also obtain

$$\frac{\left(\Xi_{v,a,d}^{k}(z)\right)'}{\Xi_{v,a,d}^{k}(z)}=\frac{\sum_{m=0}^{\infty}\frac{(-d)^{m+1}(2m+3)^{2}}{m!\Gamma_{k}(k(m+1)+v+\frac{a+1}{2}k)}z^{m}}{\sum_{m=0}^{\infty}\frac{(-d)^{m}(2m+1)^{2}}{m!\Gamma_{k}(mk+v+\frac{a+1}{2}k)}z^{m}}. \tag{2.19}$$

By comparing the coefficients in the equations (2.18) and (2.19), we obtain

$$\varpi_{1}=\frac{18d}{2v+(a+1)k},\quad \varpi_{2}=\frac{8d^{2}\left(56v+(109+28a)k\right)}{(2v+(a+1)k)^{2}(2v+(a+3)k)},$$
$$\varpi_{3}=\frac{\begin{array}{c}\left(16d^{4}\left(-236719+a\left(167710+a\left(39011+3044a\right)\right)\right)k^{3}-4\left(83855+a\left(39011+4566a\right)\right)k^{2}v\\ -4\left(39011+9132a\right)kv^{2}-24352v^{3}\end{array}}{(2v+(a+1)k)^{4}(2v+(a+3)k)^{2}(2v+(a+5)k)}.$$

Finally, by applying the Euler Rayleigh inequalities $(\varpi_{s})^{-1/s}<{}_{k}\varkappa_{\nu,a,d,1}<\varpi_{s}/\varpi_{s+1}$, we derive the bounds given in part *(a)* for $2\sqrt{k\varkappa_{\nu,a,d,1}}$.

(b) By following the same procedure as in part (a), we conclude that the radius of convexity $r^{c}\left(h_{v,a,d}^{k}\right)$ is the smallest positive root of the equation $\left(z\left(h_{v,a,d}^{k}(z)\right)'\right)'=0$. After some straightforward calculations, we obtain

$$\Delta_{v,a,d}^{k}(z)=\left(z\left(h_{v,a,d}^{k}(z)\right)'\right)'=\Gamma_{k}\left(v+\frac{a+1}{2}k\right)\sum_{m=0}^{\infty}\frac{(-d)^{m}(m+1)^{2}}{2^{2m}m!\Gamma_{k}(mk+v+\frac{a+1}{2}k)}z^{m}. \tag{2.20}$$

Moreover, since $h_{v,a,d}^{k}$ belongs to the class $\mathcal{LP}$, it follows that $\Delta_{v,a,d}^{k}$ also belongs to $\mathcal{LP}$. In other words, all zeros of the function $\Delta_{v,a,d}^{k}$ are real.

The infinite product representation of $\Delta_{v,a,d}^{k}(z)$ is given by

$$\Delta_{v,a,d}^{k}(z)=\prod_{m=1}^{\infty}\left(1-\frac{z}{{}_{k}\epsilon_{\nu,a,d,m}}\right), \tag{2.21}$$

where ${}_{k}\epsilon_{\nu,a,d,m}$ is the mth positive zero of $\Delta_{v,a,d}^{k}$.

Taking the logarithmic derivative of both sides of (2.21), we obtain

$$\frac{\left(\Delta_{v,a,d}^{k}(z)\right)'}{\Delta_{v,a,d}^{k}(z)}=-\sum_{s=0}^{\infty}\mu_{s+1}z^{s},\quad \left(|z|<_{k}\epsilon_{\nu,a,d,1}\right). \tag{2.22}$$

Let $\mu_{s}=\sum_{m=1}^{\infty}\left({}_{k}\epsilon_{\nu,a,d,m}\right)^{-s}$. Additionally, differentiating the infinite series in (2.20) yields

$$\frac{\left(\Delta^k_{v,a,d}(z)\right)'}{\Delta^k_{v,a,d}(z)} = \frac{\sum_{m=0}^{\infty} T_m z^m}{\sum_{m=0}^{\infty} S_m z^m}, \tag{2.23}$$

where

$$T_m = \frac{(-d)^{m+1}(m+2)^2}{2^{2m+2} m! \Gamma_k(k(m+1)+v+\frac{a+1}{2}k)}$$

and

$$S_m = \frac{(-d)^m (m+1)^2}{2^{2m} m! \Gamma_k(mk+v+\frac{a+1}{2}k)}.$$

By comparing the coefficients of (2.22) and (2.23), we get

$$\mu_1 = \frac{2d}{2v+(a+1)k}, \quad \mu_2 = \frac{8d^2(16v+(1-8a)k)}{(2v+(a+1)k)^2(2v+(a+3)k)},$$
$$\mu_3 = \frac{d^4\left((14997+a(8059+3a(445+27a)))k^3+2(8059+3a(890+81a))k^2v+12(445+81a)kv^2+648v^3\right)}{32(2v+(a+1)k)^4(2v+(a+3)k)^2(2v+(a+5)k)}.$$

Applying the Euler–Rayleigh inequalities $(\mu_s)^{-1/s} < {}_k\epsilon_{\nu,a,d,1} < \mu_s/\mu_{s+1}$ for $s=1$ and $s=2$, we obtain the desired result. Thus, the proof of the theorem is complete. □

Remark 2.8 Taking $k=a=d=1$ in Theorem 2.7, the results correspond to those given in [9]. If we choose $a=1$ in Theorem 2.7, the results correspond to those given in [37].

2.3 Appendix

We provide closed-form evaluations of certain multiple sums and power sums of the zeros of the function $W^k_{v,a,d}(z)$. From (1.8) and (1.10), we have

$$2^{\frac{v}{k}}\Gamma_k\left(\nu+\frac{a+1}{2}k\right) z^{\frac{-v}{2k}} W^k_{v,a,d}(\sqrt{z}) = \prod_{n=1}^{\infty}\left(1-\frac{z}{{}_k\lambda^2_{\nu,d,n}}\right) = \sum_{n=0}^{\infty}\frac{(-d)^n\Gamma_k\left(\nu+\frac{a+1}{2}k\right)}{n!4^n\Gamma_k(nk+\nu+\frac{a+1}{2}k)}z^n.$$

Thus, using Lemma 1 from [42], we obtain

$$s_n = \sum_{t=1}^{\infty}\frac{1}{\left({}_k\lambda_{\nu,d,t}\right)^{2n}},$$

and

$$\sum_{1\le t_1<t_2<\cdots<t_n} \frac{1}{\left({}_k\lambda_{\nu,d,t_1}\right)^2\left({}_k\lambda_{\nu,d,t_2}\right)^2\cdots\left({}_k\lambda_{\nu,d,t_n}\right)^2} = \frac{d^n}{n!4^n(\mu)_{n,k}},$$

where $\mu = \nu + \frac{a+1}{2}k$.

Applying Lemma 1 again from [42] with $c = -\frac{1}{4}$, we get

$$\sum_{t=1}^{\infty} \frac{4^n(-1)^{n-1}}{\left({}_k\lambda_{\nu,d,t}\right)^{2n}} = \det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \frac{d}{(\mu)_{1,k}} \\ \frac{d}{(\mu)_{1,k}} & 1 & 0 & \cdots & 0 & \frac{d^2}{(\mu)_{2,k}} \\ \frac{d^2}{2!(\mu)_{2,k}} & \frac{d}{(\mu)_{1,k}} & 1 & \cdots & 0 & \frac{d^3}{2!(\mu)_{3,k}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{d^{n-2}}{(n-2)!(\mu)_{n-2,k}} & \frac{d^{n-3}}{(n-3)!(\mu)_{n-3,k}} & \frac{d^{n-4}}{(n-4)!(\mu)_{n-4,k}} & \cdots & 1 & \frac{d^{n-1}}{(n-2)!(\mu)_{n-1,k}} \\ \frac{d^{n-1}}{(n-1)!(\mu)_{n-1,k}} & \frac{d^{n-2}}{(n-2)!(\mu)_{n-2,k}} & \frac{d^{n-3}}{(n-3)!(\mu)_{n-3,k}} & \cdots & \frac{d}{(\mu)_{1,k}} & \frac{d^n}{(n-1)!(\mu)_{n,k}} \end{pmatrix}.$$

The first four values of s_n are

$$\begin{aligned} s_1 &= \frac{d}{4\mu}, \\ s_2 &= \frac{d^2k}{4^2\prod_{j=1}^{2}(\mu)_{j,k}}, \\ s_3 &= \frac{2d^3k^2(\mu+k)_{1,k}}{4^3\prod_{j=1}^{3}(\mu)_{j,k}}, \\ s_4 &= \frac{d^4k^3(5\mu+6k)(\mu+k)_{2,k}}{4^4\prod_{j=1}^{4}(\mu)_{j,k}}. \end{aligned}$$

Taking $k = a = d = 1$, we obtain the following sums related to the classical Bessel functions $J_\nu(z)$

$$\begin{aligned} s_1 &= \sum_{t=1}^{\infty}\frac{1}{j_{\nu,t}^2} = \frac{1}{4(v+1)}, \\ s_2 &= \sum_{t=1}^{\infty}\frac{1}{j_{\nu,t}^4} = \frac{1}{4^2\prod_{j=1}^{2}(v+1)_j}, \\ s_3 &= \sum_{t=1}^{\infty}\frac{1}{j_{\nu,t}^6} = \frac{2(v+2)_1}{4^3\prod_{j=1}^{3}(v+1)_j}, \\ s_4 &= \sum_{t=1}^{\infty}\frac{1}{j_{\nu,t}^8} = \frac{(5v+11)(v+2)_2}{4^4\prod_{j=1}^{4}(v+1)_j}, \end{aligned}$$

where $j_{\nu,t}$ is the tth positive zero of $J_\nu(z)$.

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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