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ORIGINAL RESEARCH

Neutrosophic hesitant fuzzy soft sets: algebraic structure, topology, and application in decision making

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Abstract
This study introduces a novel structure called the Neutrosophic Hesitant Fuzzy Soft Set (NHFSS), which integrates the advantages of neutrosophic sets, hesitant fuzzy sets, and soft set theory to provide a richer framework for modeling complex uncertainty in decision-making problems. After formally defining the algebraic operations for NHFSS, such as point, union, intersection, complement, and difference, we propose a new class of topological spaces based on this structure. The topological framework extends classical concepts like open sets, closed set, interior and closure into the neutrosophic hesitant fuzzy soft environment. To demonstrate the practicality of the proposed model, a multi-criteria decision-making problem is presented concerning academic staff evaluation. The effectiveness of the approach is compared with other neutrosophic decision-making techniques, and its flexibility in capturing indeterminacy and hesitation is discussed in depth. This work contributes both a theoretical foundation and an application framework for future studies on soft computing and uncertain systems.

Keywords Neutrosophic hesitant fuzzy soft set · Neutrosophic hesitant fuzzy soft topological space · Multi-criteria decision making · Uncertainty modeling · Hesitant fuzzy point · Hesitant fuzzy logic

Mathematics Subject Classification 03E72 · 54A40 · 90B50 · 68T37

1 Introduction

In mathematical modeling and decision analysis, managing uncertainty is an essential component. Classical set theory, with its binary nature, is often insufficient in capturing the nuanced and vague characteristics of real-world phenomena. To address this limitation, Zadeh introduced the concept of fuzzy sets in 1965 (Zadeh 1965), wherein an element's membership to a set is expressed by a degree within the interval [0, 1]. This development laid the foundation for fuzzy logic, fuzzy topology, and approximate reasoning, influencing a vast body of research in control systems, pattern recognition, and decision-making.

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However, fuzzy sets alone were not flexible enough to handle the diversity and granularity of uncertainty observed in complex systems. In response, Molodtsov introduced soft set theory in 1999 (Molodtsov 1999), a parameterized framework that associates each parameter with an approximate description of an object. Soft sets provide a broader, non-restrictive model for handling indeterminate information, especially when the dependency between parameters and objects must be captured explicitly. Over the years, this model has been extended and applied in various fields, including economics, engineering, and medical diagnosis (Asghar et al. 2024; Rahman et al. 2025; Kong et al. 2011; Maji et al. 2002; Yang et al. 2013; Yolcu et al. 2024).

Furthermore, several recent and foundational developments in vague soft set theory have enriched the modeling of uncertainty and expert-based evaluations. Early work by Alhazaymeh and Hassan (2017) introduced the vague soft expert set for structured decision contexts, while (Tahat et al. 2015) examined ordering mechanisms within vague soft frameworks. More recent contributions include the transitive closure of vague soft set relations (Al-Qudah

et al. 2022) and the Fermatean vague soft set model for enhanced decision-making environments (Al-shboul et al. 2024). These studies provide additional extensions of soft set theory that complement the development of hybrid models such as our NHFSS framework.

Meanwhile, to incorporate hesitation and partial preferences more explicitly, Torra developed the theory of hesitant fuzzy sets (HFS) in 2010 (Torra 2010). Unlike classical fuzzy sets, HFS allows a set of possible membership degrees for each element, thereby accommodating hesitation among several values (Faizi et al. 2018; Qiyas et al. 2025). Hesitant fuzzy soft sets (HFSS), a merger of soft set theory and HFS, were proposed to handle parameterized hesitation more efficiently and found successful applications in multi-criteria decision-making, particularly in environments involving multiple experts and conflicting judgments (Babitha and John 2013; Stojanovic et al. 2025; Suo et al. 2021).

Simultaneously, Smarandache introduced neutrosophic set theory (Smarandache 2005), aiming to generalize fuzzy and intuitionistic fuzzy sets. A neutrosophic set characterizes each element by three independent membership degrees: truth-membership, indeterminacy-membership, and falsity-membership, all of which are allowed to vary independently within [0, 1]. This three-component system provides a much more flexible and nuanced approach for modeling uncertain, inconsistent, and incomplete information (Majumdar 2015; Shahzadi et al. 2017).

Building on neutrosophic theory, researchers developed neutrosophic soft sets (NSS) (Maji et al. 2015; Ozturk et al. 2019), combining the parameter-based flexibility of soft sets with the expressive power of neutrosophic sets. NSS offers enhanced capability for handling real-world problems where data incompleteness and parameter uncertainty are simultaneously present. Following these developments, the integration of neutrosophic logic into topological structures became a compelling area of study (Rahman et al. 2025; Yolcu and Ozturk 2025; Yolcu et al. 2023; Yolcu and Aka 2023; Yolcu et al. 2021).

Neutrosophic topological spaces (NTS) were introduced to redefine the classical topological concepts such as open sets, closure, interior, and neighborhood systems within the neutrosophic context. Shabir and Naz (2011) initiated foundational work in this domain. Their framework was extended by subsequent studies to include continuity, compactness, and connectedness in neutrosophic environments. These structures serve as a basis for studying convergence and continuity in uncertain environments.

Naz and Shabir (2014) proposed neutrosophic soft topological spaces in 2014. They merged neutrosophic soft sets with topological axioms, such as open and closed sets, interiors, and closures. Their work laid the theoretical groundwork for defining neutrosophic soft continuous functions, base elements, and neighborhood systems. Later, Ozturk et al. redefined it and examined its various properties (Gunduz et al. 2019; Ozturk et al. 2019).

In parallel, researchers sought to merge the strengths of hesitant fuzzy and neutrosophic sets, resulting in the concept of Neutrosophic Hesitant Fuzzy Sets (NHFS) (Riaz et al. 2022; Sahin and Liu 2017; Ye 2015). These sets allow multiple values for T, I, and F degrees, effectively capturing layered uncertainty. Gao et al. (2022) proposed algorithms under this paradigm, and further generalizations such as Single-Valued Neutrosophic Hesitant Fuzzy Sets (SVNHFS) were explored for computational feasibility and application in decision-making contexts (Wang and Li 2018). Additionally, recent works have aimed to construct topological structures under SVNHFS, leading to the definition of SVNHFS-topologies, neighborhood systems, and continuity concepts tailored to such frameworks.

The primary motivation for introducing the Neutrosophic Hesitant Fuzzy Soft Set (NHFSS) framework is to unify three complementary uncertainty-handling mechanisms that existing models address only in isolation. Neutrosophic sets effectively represent truth, indeterminacy, and falsity degrees, but cannot model hesitation. Hesitant fuzzy sets incorporate hesitation but ignore parameter dependency and multi-dimensional uncertainty. Soft sets handle parameterized vagueness yet lack the ability to express simultaneous neutrosophic and hesitant characteristics. Real-world decision-making problems especially those involving conflicting expert opinions, incomplete data, and multi-parameter evaluations require a structure that combines all three dimensions. Therefore, NHFSS is proposed as a comprehensive mathematical environment capable of representing layered uncertainty. This unified structure fills a significant gap in the literature by offering a more expressive, flexible, and realistic modeling tool for both algebraic analysis and decision-making applications.

In this study, we contribute to this growing body of research by proposing a new structure: the Neutrosophic Hesitant Fuzzy Soft Set (NHFSS). This model integrates hesitant fuzzy logic, neutrosophic components, and soft parameterization into a unified system, thereby offering a rich mathematical structure capable of expressing complex real-world uncertainties. Subsequently, we introduce NHFSS-based topological spaces and investigate their properties. Finally, the proposed theory is applied to a practical decision-making scenario involving academic staff evaluation, demonstrating its effectiveness, adaptability, and relevance in uncertain and parameter-dependent environments.

2 Preliminaries

In this section, we will give some preliminary information for the present study.

Definition 1 (Smarandache 2005) A neutrosophic set A on the universe of discourse $\aleph$ is defined as:

$$A = \{\langle \pi, \widetilde{t}_A(\pi), \widetilde{i}_A(\pi), \widetilde{f}_A(\pi)\rangle : \pi \in \aleph\},$$

where $\widetilde{t}, \widetilde{i}, \widetilde{f} : \aleph \to]^-0, 1^+[$ and $^-0 \leq \widetilde{t}_A(\pi) + \widetilde{i}_A(\pi) + \widetilde{f}_A(\pi) \leq 3^+$.

Definition 2 (Molodtsov 1999) Let $\aleph$ be an initial universe, $\Re$ be a set of all parameters and $SP(\aleph)$ denotes the power set of $\aleph$. A pair $(\widetilde{f}, \Re)$ is called a soft set over $\aleph$, where $\widetilde{f}$ is a mapping given by $\widetilde{f} : \Re \to SP(\aleph)$.

In other words, the soft set is a parameterized family of subsets of the set $\aleph$. For $r \in \Re$, $\widetilde{f}(r)$ may be considered as the set of r-elements of the soft set $(\widetilde{f}, \Re)$, or as the set of r -approximate elements of the soft set, i.e.,

$$(\widetilde{f}, \Re) = \left\{(r, \widetilde{f}(r)) : r \in \Re,\ \widetilde{f} : \Re \to SP(\aleph)\right\}.$$

Definition 3 (Torra 2010) Let $\aleph$ be a universe of discourse. A hesitant fuzzy set H on $\aleph$ is defined in terms of a fuction h_H that when applied to π returns a non-empty finite subset of [0, 1].

$$H = \{\langle \pi, h_H(\pi)\rangle | \pi \in \aleph\}$$

where $h_H(\pi)$ denotes the possible membership degrees of the element $\pi \in \aleph$ to the set H.

Definition 4 (Maji et al. 2015) Let $\aleph$ be an initial universe set and $\Re$ be a set of parameters. Let $NSP(\aleph)$ denote the set of all neutrosophic sets of $\aleph$. Then, a neutrosophic soft set $(\widetilde{\delta}, \Re)$ over $\aleph$ is a set defined by a set valued function $\widetilde{\delta}$ representing a mapping $\widetilde{\delta} : \Re \to NHFP(\aleph)$ where $\widetilde{\delta}$ is called approximate function of the neutrosophic soft set $(\widetilde{\delta}, \Re)$. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $NSP(\aleph)$ and therefore it can be written as a set of ordered pairs,

$$(\widetilde{\delta}, \Re) = \left\{\left(r, \left\langle \pi, \widetilde{t}_{\widetilde{\delta}(r)}(\pi), \widetilde{i}_{\widetilde{\delta}(r)}(\pi), \widetilde{f}_{\widetilde{\delta}(r)}(\pi)\right\rangle : \pi \in \aleph\right) : r \in \Re\right\}$$

where $\widetilde{t}_{\widetilde{\delta}(r)}(\pi), \widetilde{i}^{(j)}_{\widetilde{\delta}(r)}(\pi), \widetilde{f}_{\widetilde{\delta}(r)}(\pi) \in [0, 1]$, respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $\widetilde{f}(r)$. Since supremum of each $\widetilde{t}, \widetilde{i}, \widetilde{f}$ is 1 so the inequality $0 \leq \widetilde{t}_{\widetilde{\delta}(r)}(\pi) + \widetilde{i}_{\widetilde{\delta}(r)}(\pi) + \widetilde{f}_{\widetilde{\delta}(r)}(\pi) \leq 3$ is obvious.

Definition 5 (Ye 2015) Let $\aleph$ be a universe of discourse. Single-Valued Neutrosophic Hesitant Fuzzy Set (SVNHFS) A in $\aleph$ is defined as a set of ordered triples:

$$A = \{\langle \pi, T_A(\pi), I_A(\pi), F_A(\pi)\rangle \mid \pi \in \aleph\}$$

where

$$t_A(\pi), i_A(\pi), f_A(\pi) \subseteq [0, 1]$$

are all non-empty finite subsets of [0, 1] representing the sets of possible: $t_A(\pi)$: truth-membership degrees, $i_A(\pi)$: indeterminacy-membership degrees, $f_A(\pi)$: falsity-membership degrees.

This structure allows for the expression of hesitation or uncertainty not only in the truth-membership but also in the indeterminacy and falsity assessments of each element.

3 Neutrosophic hesitant fuzzy soft sets

Definition 6 Let $\aleph$ be an initial universe set and $\Re$ be a set of parameters. Let $NHFP(\aleph)$ denote the set of all neutrosophic hesitant fuzzy sets of $\aleph$. Then, a neutrosophic hesitant fuzzy soft set $nhf(\widetilde{\delta}, \Re)$ over $\aleph$ is a set defined by a set valued function $\widetilde{\delta}$ representing a mapping $\widetilde{\delta} : \Re \to NHFP(\aleph)$ where $\widetilde{\delta}$ is called approximate function of the neutrosophic hesitant fuzzy soft set $nhf(\widetilde{\delta}, \Re)$. In other words, the neutrosophic hesitant fuzzy soft set is a parameterized family of some elements of the set $NHFP(\aleph)$ and therefore it can be written as a set of ordered pairs,

$$nhf(\widetilde{\delta}, \Re) = \left\{\left(r, \left\langle \pi, \widetilde{t}(\pi), \widetilde{i}(\pi), \widetilde{f}(\pi)\right\rangle : \pi \in \aleph\right) : r \in \Re\right\}$$

where $\widetilde{t}(\pi) = \{\alpha_i : i = 1, ..., m\} \subseteq [0, 1]$, $\widetilde{i}(\pi) = \{\beta_j : j = 1, ..., p\} \subseteq [0, 1]$, $\widetilde{f}(\pi) = \{\gamma_k : k = 1, ..., q\} \subseteq [0, 1]$ are *finite (nonempty) hesitant subsets* of [0, 1] representing, respectively, the truth, indeterminacy, and falsity–membership degrees. The cardinalities $m, p, q \in \mathbb{N}$ are arbitrary and may differ from one component to another

(t vs. i vs. f). $NHFSS(\aleph,\Re)$ denote the set of all neutrosophic hesitant fuzzy soft sets of $(\aleph,\Re)$.

Example 1 Let $\aleph=\{\pi_1,\pi_2,\pi_3,\pi_4\}$ and $\Re=\{r_1,r_2\}$. Then, a neutrosophic hesitant fuzzy soft set $nhf(\widetilde{\delta},\Re)$ can be expressed as follows:

$$nhf(\widetilde{\delta},\Re)=\left\{\begin{array}{c} r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.5,0.7\},\{0.7,0.2,0.6\},\{0.6,0.5\}\rangle,\\ \langle\pi_2,\{0.4\},\{0.3,0.4\},\{0.8,0.5,0.6,0.2\}\rangle,\\ \langle\pi_3,\{0.3,0.2\},\{0.3\},\{0.7,0.8\}\rangle\end{array}\right\}\\ r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.4,0.7\},\{0.6,0.8,0.6\},\{0.8,0.5\}\rangle,\\ \langle\pi_2,\{0.3,0.1\},\{0.7\},\{0.7,0.2\}\rangle,\\ \langle\pi_3,\{0.3,0.1\},\{0.2,0.3\},\{0.4,0.3,0.7\}\rangle,\\ \langle\pi_4,\{0.4,0.1\},\{0.2,0.4\},\{0.9,0.4,0.2\}\rangle\end{array}\right\}\end{array}\right\}$$

Definition 7 A neutrosophic hesitant fuzzy soft set $nhf(\widetilde{\delta},\Re)$ is said to be null neutrosophic hesitant fuzzy soft set if $\widetilde{t}^{(i)}_{\widetilde{\delta}(r)}(\pi)=0$, $\widetilde{i}^{(j)}_{\widetilde{\delta}(r)}(\pi)=0$, $\widetilde{f}^{(k)}_{\widetilde{\delta}(r)}(\pi)=1$; $\forall r\in\Re,\forall\pi\in\aleph$. The null neutrosophic hesitant fuzzy soft set is denoted by $nhf(\tilde{0}_{(\aleph,\Re)})$. A neutrosophic hesitant fuzzy soft set $nhf(\widetilde{\delta},\Re)$ is said to be absolute neutrosophic hesitant fuzzy soft set if $\widetilde{t}^{(i)}_{\widetilde{\delta}(r)}(\pi)=1$, $\widetilde{i}^{(j)}_{\widetilde{\delta}(r)}(\pi)=1$, $\widetilde{f}^{(k)}_{\widetilde{\delta}(r)}(\pi)=0$; $\forall r\in\Re,\forall\pi\in\aleph$.The null neutrosophic hesitant fuzzy soft set is denoted by $nhf(\tilde{1}_{(\aleph,\Re)})$.

Definition 8 Let us consider two neutrosophic hesitant fuzzy soft sets $nhf(\widetilde{\delta}_1,\Re)$ and $nhf(\widetilde{\delta}_2,\Re)$ over the universe set $\aleph$. Some operations on neutrosophic hesitant fuzzy soft sets are defined as follows

1. The complement of $(\widetilde{\delta},\Re)$ is denoted by $(\widetilde{\delta},\Re)^c$ and is defined by $(\widetilde{\delta},\Re)^c=\{(\langle\pi,\widetilde{f}^{(k)}_{\widetilde{\delta}(r)}(\pi),\ 1-\widetilde{i}^{(j)}_{\widetilde{\delta}(r)}(\pi),\ \widetilde{t}^{(i)}_{\widetilde{\delta}(r)}(\pi)\rangle:\pi\in\aleph):r\in\Re\}$.
2. $nhf(\widetilde{\delta}_1,\Re)$ is said to be neutrosophic hesitant fuzzy soft subset of $nhf(\widetilde{\delta}_2,\Re)$ if $\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi)\le\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)$, $\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi)\le\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r)}(\pi)$, $\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi)\ge\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)$, $\forall r\in\Re$, $\forall\pi\in\aleph$. It is denoted by $nhf(\widetilde{\delta}_1,\Re)\ \widetilde{\subseteq} nhf(\widetilde{\delta}_2,\Re)$. .$nhf(\widetilde{\delta}_1,\Re)$ is said to be neutrosophic hesitant fuzzy soft equal to $nhf(\widetilde{\delta}_2,\Re)$ if $nhf(\widetilde{\delta}_1,\Re)$ is neutrosophic hesitant fuzzy soft subset of $nhf(\widetilde{\delta}_2,\Re)$ and $nhf(\widetilde{\delta}_2,\Re)$ is neutrosophic hesitant fuzzy soft subset of $nhf(\widetilde{\delta}_1,\Re)$. It is denoted by $nhf(\widetilde{\delta}_1,\Re)=nhf(\widetilde{\delta}_2,\Re)$.
3. The union of two neutrosophic hesitant fuzzy soft sets is defined as follows:

$$nhf(\widetilde{\delta}_1,\Re)\cup nhf(\widetilde{\delta}_2,\Re)=nhf(\widetilde{\delta}_3,\Re)$$
$$=\left\{\left(r,\left\langle\begin{array}{c}\pi,\max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \max\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right\rangle:\pi\in\aleph\right):r\in\Re\right\}$$

4. The intersection of two neutrosophic hesitant fuzzy soft sets is defined as follows:

$$nhf(\widetilde{\delta}_1,\Re)\cap nhf(\widetilde{\delta}_2,\Re)=nhf(\widetilde{\delta}_3,\Re)$$
$$=\left\{\left(r,\left\langle\begin{array}{c}\pi,\min\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \min\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \max\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right\rangle:\pi\in\aleph\right):r\in\Re\right\}$$

5. The difference of two neutrosophic hesitant fuzzy soft sets is defined as follows: $nhf(\widetilde{\delta}_1,\Re)\backslash nhf(\widetilde{\delta}_2,\Re)=nhf(\widetilde{\delta}_3,\Re)$ and is defined by

$$nhf(\widetilde{\delta}_3,\Re)=nhf(\widetilde{\delta}_1,\Re)\cap nhf(\widetilde{\delta}_2,\Re)^c$$
$$=\left\{\left(r,\left\langle\begin{array}{c}\pi,\min\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \min\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi),1-\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \max\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right\rangle:\pi\in\aleph\right):r\in\Re\right\}$$

6. Let $\{(\widetilde{\delta}_i,\Re)|i\in\widetilde{i}\}$ be a family of neutrosophic hesitant fuzzy soft sets over the universe set $\aleph$. Then

$$\bigcup_{i\in\widetilde{i}}(\widetilde{\delta}_i,\Re)$$
$$=\left\{\left(r,\left\langle\pi,\sup\left[\widetilde{t}^{(i)}_{\widetilde{\delta}_i(r)}(\pi)\right]_{i\in\widetilde{i}},\sup\left[\widetilde{i}^{(j)}_{\widetilde{\delta}_i(r)}(\pi)\right]_{i\in\widetilde{i}},\right.\right.\right.$$
$$\left.\left.\left.\inf\left[\widetilde{f}^{(k)}_{\widetilde{\delta}_i(r)}(\pi)\right]_{i\in\widetilde{i}}\right\rangle:\pi\in\aleph\right):r\in\Re\right\},$$
$$\bigcap_{i\in\widetilde{i}}(\widetilde{\delta}_i,\Re)$$
$$=\left\{\left(r,\left\langle\pi,\inf\left[\widetilde{t}^{(i)}_{\widetilde{\delta}_i(r)}(\pi)\right]_{i\in\widetilde{i}},\inf\left[\widetilde{i}^{(j)}_{\widetilde{\delta}_i(r)}(\pi)\right]_{i\in\widetilde{i}},\right.\right.\right.$$
$$\left.\left.\left.\sup\left[\widetilde{f}^{(k)}_{\widetilde{\delta}_i(r)}(\pi)\right]_{i\in\widetilde{i}}\right\rangle:\pi\in\aleph\right):r\in\Re\right\}.$$

7. The "AND" operation of two neutrosophic hesitant fuzzy soft sets is defined as follows: $nhf(\widetilde{\delta}_1,\Re)\wedge nhf(\widetilde{\delta}_2,\Re)=(\widetilde{\delta}_3,\Re\times\Re)=$

$$\left\{\left(\begin{array}{c}\pi, \min\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi), \widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\}, \\ (r_1, r_2), \left\langle \min\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi), \widetilde{i}^{(j)}_{\widetilde{\delta}_2(r)}(\pi)\right\}, \right\rangle : \pi \in \aleph \\ \max\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi), \widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right) : (r_1, r_2) \in \Re \times \Re\right\}$$

8. The "OR" operation of two neutrosophic hesitant fuzzy soft sets is defined as follows: $nhf(\widetilde{\delta}_1, \Re) \vee nhf(\widetilde{\delta}_2, \Re) = (\widetilde{\delta}_3, \Re \times \Re) =$

$$\left\{\left(\begin{array}{c}\pi, \max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi), \widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\}, \\ (r_1, r_2), \left\langle \max\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi), \widetilde{i}^{(j)}_{\widetilde{\delta}_2(r)}(\pi)\right\}, \right\rangle : \pi \in \aleph \\ \min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi), \widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right) : (r_1, r_2) \in \Re \times \Re\right\}$$

Remark 1 During the operations of two neutrosophic hesitant fuzzy soft sets, if the number of truth-membership, indeterminacy-membership, or falsity-membership values associated with a given parameter-alternative pair is unequal between the two sets, the missing values are compensated by padding with default elements to enable element-wise operations. Specifically, missing truth and indeterminacy values are padded with 0 (indicating no degree of truth or indeterminacy), while missing falsity values are padded with 1 (indicating full falsity by default). This approach ensures that the max operator for truth and indeterminacy degrees, and the min operator for falsity degrees, can be applied uniformly across all components of the hesitant sets

Theorem 1 *Let $nhf(\tilde{\delta}_1, \Re)$ and $nhf(\tilde{\delta}_2, \Re)$ be two neutrosophic hesitant fuzzy soft sets over the universe $\aleph$. Then, their union $nhf(\tilde{\delta}_1, \Re) \cup nhf(\tilde{\delta}_2, \Re)$ and intersection $nhf(\tilde{\delta}_1, \Re) \cap nhf(\tilde{\delta}_2, \Re)$ are also neutrosophic hesitant fuzzy soft sets.*

Proof Let $r \in \Re$ and $\pi \in \aleph$. Let $\tilde{t}_{\tilde{\delta}_1(r)}(\pi)$ and $\tilde{t}_{\tilde{\delta}_2(r)}(\pi)$ denote the sets of truth-membership values for the two sets, respectively. By Definition 6, these are finite subsets of [0, 1].According to the union operation defined in Definition 8, the truth-membership set of the union is formed by the maximum operation (after necessary padding):

$$\tilde{t}_{\cup}(\pi) = \{\max(\alpha_1, \alpha_2) : \alpha_1 \in \tilde{t}_{\tilde{\delta}_1(r)}(\pi), \alpha_2 \in \tilde{t}_{\tilde{\delta}_2(r)}(\pi)\}$$

Since $\alpha_1, \alpha_2 \in [0, 1]$, it follows that $\max(\alpha_1, \alpha_2) \in [0, 1]$. Thus, $\tilde{t}_{\cup}(\pi)$ remains a finite subset of [0, 1].Similarly, the indeterminacy (using max) and falsity (using min) membership sets for the union are:

$$\tilde{i}_{\cup}(\pi) = \{\max(\beta_1, \beta_2) : \beta_1 \in \tilde{i}_{\tilde{\delta}_1(r)}(\pi), \beta_2 \in \tilde{i}_{\tilde{\delta}_2(r)}(\pi)\} \subseteq [0, 1]$$

$$\tilde{f}_{\cup}(\pi) = \{\min(\gamma_1, \gamma_2) : \gamma_1 \in \tilde{f}_{\tilde{\delta}_1(r)}(\pi), \gamma_2 \in \tilde{f}_{\tilde{\delta}_2(r)}(\pi)\} \subseteq [0, 1]$$

Since all resulting membership sets are subsets of [0, 1], the closure property holds for the union. The proof for the intersection follows the same logic using the corresponding operators. □

Proposition 2 *Let $nhf(\widetilde{\delta}_1, \Re)$, $nhf(\widetilde{\delta}_2, \Re)$ and $nhf(\widetilde{\delta}_3, \Re)$ be neutrosophic hesitant fuzzy soft sets over the universe set $\aleph$. Then,*

1. $nhf(\widetilde{\delta}_1, \Re) \cup [nhf(\widetilde{\delta}_2, \Re) \cup nhf(\widetilde{\delta}_3, \Re)]$ $= [nhf(\widetilde{\delta}_1, \Re) \cup nhf(\widetilde{\delta}_2, \Re)] \cup nhf(\widetilde{\delta}_3, \Re)$ *and* $nhf(\widetilde{\delta}_1, \Re) \cap [nhf(\widetilde{\delta}_2, \Re) \cap nhf(\widetilde{\delta}_3, \Re)]$ $= [nhf(\widetilde{\delta}_1, \Re) \cap nhf(\widetilde{\delta}_2, \Re)] \cap nhf(\widetilde{\delta}_3, \Re)$;
2. $nhf(\widetilde{\delta}_1, \Re) \cup [nhf(\widetilde{\delta}_2, \Re) \cap nhf(\widetilde{\delta}_3, \Re)]$ $= [nhf(\widetilde{\delta}_1, \Re) \cup nhf(\widetilde{\delta}_2, \Re)] \cap [nhf(\widetilde{\delta}_1, \Re)$ $\cup nhf(\widetilde{\delta}_3, \Re)]$ *and* $nhf(\widetilde{\delta}_1, \Re) \cap [nhf(\widetilde{\delta}_2, \Re)$ $\cup nhf(\widetilde{\delta}_3, \Re)] = [nhf(\widetilde{\delta}_1, \Re) \cap nhf(\widetilde{\delta}_2, \Re)]$ $\cup [nhf(\widetilde{\delta}_1, \Re) \cap nhf(\widetilde{\delta}_3, \Re)]$;
3. $nhf(\widetilde{\delta}_1, \Re) \cup nhf(\tilde{0}_{(\aleph, \Re)}) = nhf(\widetilde{\delta}_1, \Re)$ *and* $nhf(\widetilde{\delta}_1, \Re) \cap nhf(\tilde{0}_{(\aleph, \Re)}) = nhf(\tilde{0}_{(\aleph, \Re)})$;
4. $nhf(\widetilde{\delta}_1, \Re) \cup 1_{(\aleph, \Re)} = 1_{(\aleph, \Re)}$ *and* $nhf(\widetilde{\delta}_1, \Re) \cap 1_{(\aleph, \Re)}$ $= nhf(\widetilde{\delta}_1, \Re)$.

Proof Straightforward. □

Proposition 3 *Let $nhf(\widetilde{\delta}_1, \Re)$ and $nhf(\widetilde{\delta}_2, \Re)$ be two neutrosophic hesitant fuzzy soft sets over the universe set $\aleph$. Then,*

1. $[nhf(\widetilde{\delta}_1,\Re)\cup nhf(\widetilde{\delta}_2,\Re)]^c = nhf(\widetilde{\delta}_1,\Re)^c\cap nhf(\widetilde{\delta}_2,\Re)^c$;
2. $[nhf(\widetilde{\delta}_1,\Re)\cap nhf(\widetilde{\delta}_2,\Re)]^c$ $nhf(\widetilde{\delta}_1,\Re)^c\cup nhf(\widetilde{\delta}_2,\Re)^c$.

Proof 1. For all $r\in\Re$ and $\pi\in\aleph$,

$$nhf(\widetilde{\delta}_1,\Re)\cup nhf(\widetilde{\delta}_2,\Re)$$
$$=\left\{\left\langle \begin{array}{c}\pi,\max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\},\max\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{i}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right\rangle\right\}$$
$$\left[nhf(\widetilde{\delta}_1,\Re)\cup nhf(\widetilde{\delta}_2,\Re)\right]^c$$
$$=\left\{\left\langle \begin{array}{c}\pi,\min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\},1-\max\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{i}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right\rangle\right\}.$$

Now,

$$nhf(\widetilde{\delta}_1,\Re)^c=\left\{\left\langle \pi,\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi),1-\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi)\right\rangle\right\},$$
$$nhf(\widetilde{\delta}_2,\Re)^c=\left\{\left\langle \pi,\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi),1-\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\rangle\right\}.$$

Then,

$$nhf(\widetilde{\delta}_1,\Re)^c\cap nhf(\widetilde{\delta}_2,\Re)^c$$
$$=\left\{\left\langle \begin{array}{c}\pi,\min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\},\min\left\{\left(1-\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi)\right),\left(1-\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r)}(\pi)\right)\right\},\\ \max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right\rangle\right\}$$
$$=\left\{\left\langle \begin{array}{c}\pi,\min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r)}(\pi)\right\},1-\max\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{i}_{\widetilde{\delta}_2(r)}(\pi)\right\},\\ \max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r)}(\pi)\right\}\end{array}\right\rangle\right\}.$$

Therefore, $[nhf(\widetilde{\delta}_1,\Re)\cup nhf(\widetilde{\delta}_2,\Re)]^c = nhf(\widetilde{\delta}_1,\Re)^c$ $nhf(\widetilde{\delta}_2,\Re)^c$.

2. It is obtained in a similar way. □

Proposition 4 *Let $nhf(\widetilde{\delta}_1,\Re)$ and $nhf(\widetilde{\delta}_2,\Re)$ be two neutrosophic hesitant fuzzy soft sets over the universe set $\aleph$. Then,*

1. $[nhf(\widetilde{\delta}_1,\Re)\vee nhf(\widetilde{\delta}_2,\Re)]^c = nhf(\widetilde{\delta}_1,\Re)^c$ $\wedge nhf(\widetilde{\delta}_2,\Re)^c$;
2. $[nhf(\widetilde{\delta}_1,\Re)\wedge nhf(\widetilde{\delta}_2,\Re)]^c = nhf(\widetilde{\delta}_1,\Re)^c$ $\vee nhf(\widetilde{\delta}_2,\Re)^c$.

Proof 1. For all $(r_1,r_2)\in\Re\times\Re$ and $\pi\in\aleph$,

$$nhf(\widetilde{\delta}_1,\Re)\vee nhf(\widetilde{\delta}_2,\Re)$$
$$=\left\{\left\langle \begin{array}{c}\pi,\max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{t}^{(i)}_{\widetilde{\delta}_2(r_2)}(\pi)\right\},\max\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r_2)}(\pi)\right\},\\ \min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{f}_{\widetilde{\delta}_2(r_2)}(\pi)\right\}\end{array}\right\rangle\right\},$$
$$\left[nhf(\widetilde{\delta}_1,\Re)\vee nhf(\widetilde{\delta}_2,\Re)\right]^c$$
$$=\left\{\left\langle \begin{array}{c}\pi,\min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r_2)}(\pi)\right\},1-\max\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r_2)}(\pi)\right\},\\ \max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{t}_{\widetilde{\delta}_2(r_2)}(\pi)\right\}\end{array}\right\rangle\right\}.$$

On the other hand,

$$nhf(\widetilde{\delta}_1,\Re)^c=\left\{\left\langle \pi,\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r_1)}(\pi),1-\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{t}_{\widetilde{\delta}_1(r_1)}(\pi)\right\rangle : r_1\in\Re\right\},$$
$$nhf(\widetilde{\delta}_2,\Re)^c=\left\{\left\langle \pi,\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r_2)}(\pi),1-\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r_2)}(\pi),\widetilde{t}_{\widetilde{\delta}_2(r_2)}(\pi)\right\rangle : r_2\in\Re\right\}.$$

Then,

$$nhf(\widetilde{\delta}_1,\Re)^c\wedge nhf(\widetilde{\delta}_2,\Re)^c$$
$$=\left\{\left\langle \begin{array}{c}\pi,\min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r_2)}(\pi)\right\},\min\left\{\left(1-\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r_1)}(\pi)\right),\left(1-\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r_2)}(\pi)\right)\right\},\\ \max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{t}_{\widetilde{\delta}_2(r_2)}(\pi)\right\}\end{array}\right\rangle\right\}$$
$$=\left\{\left\langle \begin{array}{c}\pi,\min\left\{\widetilde{f}^{(k)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{f}^{(k)}_{\widetilde{\delta}_2(r_2)}(\pi)\right\},1-\max\left\{\widetilde{i}^{(j)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{i}^{(j)}_{\widetilde{\delta}_2(r_2)}(\pi)\right\},\\ \max\left\{\widetilde{t}^{(i)}_{\widetilde{\delta}_1(r_1)}(\pi),\widetilde{t}_{\widetilde{\delta}_2(r_2)}(\pi)\right\}\end{array}\right\rangle\right\}.$$

Hence, $[nhf(\widetilde{\delta}_1,\Re)\vee nhf(\widetilde{\delta}_2,\Re)]^c = nhf(\widetilde{\delta}_1,\Re)^c$ $\wedge nhf(\widetilde{\delta}_2,\Re)^c$. □

Example 2 Let $\aleph=\{\pi_1,\pi_2,\pi_3,\pi_4\}$ and $\Re=\{r_1,r_2\}$. Then, a neutrosophic hesitant fuzzy soft sets $nhf(\widetilde{\delta}_1,\Re)$ and $nhf(\widetilde{\delta}_2,\Re)$ can be expressed as follows:

$$nhf(\widetilde{\delta}_1,\Re)=\left\{\begin{array}{c} r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.5,0.7\},\{0.7,0.2,0.6\},\{0.6,0.5\}\rangle,\\ \langle\pi_2,\{0.4\},\{0.3,0.4\},\{0.8,0.5,0.6,0.2\}\rangle,\\ \langle\pi_3,\{0.3,0.2\},\{0.3\},\{0.7,0.8\}\rangle\end{array}\right\}\\ r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.4,0.7\},\{0.6,0.8,0.6\},\{0.8,0.5\}\rangle,\\ \langle\pi_2,\{0.3,0.1\},\{0.7\},\{0.7,0.2\}\rangle,\\ \langle\pi_3,\{0.3,0.1\},\{0.2,0.3\},\{0.4,0.3,0.7\}\rangle,\\ \langle\pi_4,\{0.4,0.1\},\{0.2,0.4\},\{0.9,0.4,0.2\}\rangle\end{array}\right\}\end{array}\right\}$$
$$nhf(\widetilde{\delta}_2,\Re)=\left\{\begin{array}{c} r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.2,0.7\},\{0.2,0.3\},\{0.4,0.6\}\rangle,\\ \langle\pi_3,\{0.5,0.1\},\{0.1\},\{0.2,0.4\}\rangle,\\ \langle\pi_4,\{0.7,0.4,0.6\},\{0.5,0.8\},\{0.7,0.4\}\rangle\end{array}\right\}\\ r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.2,0.8,0.3\},\{0.5,0.4,0.3\},\{0.5,0.5\}\rangle,\\ \langle\pi_2,\{0.7\},\{0.3,0.4\},\{0.2,0.1,0.2\}\rangle,\\ \langle\pi_3,\{0.5,0.6,0.2,0.1\},\{0.4,0.7,0.3\},\{0.2,0.5,0.6\}\rangle,\\ \langle\pi_4,\{0.1,0.3\},\{0.4,0.5\},\{0.7,0.2,0.9\}\rangle\end{array}\right\}\end{array}\right\}$$

Then,

$$nhf(\widetilde{\delta}_1,\Re)\cup nhf(\widetilde{\delta}_2,\Re)$$
$$=\left\{\begin{array}{c} r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.5,0.7\},\{0.7,0.3,0.6\},\{0.4,0.5\}\rangle,\\ \langle\pi_2,\{0.4\},\{0.3,0.4\},\{0.8,0.5,0.6,0.2\}\rangle,\\ \langle\pi_3,\{0.5,0.2\},\{0.3\},\{0.2,0.4\}\rangle,\\ \langle\pi_4,\{0.7,0.4,0.6\},\{0.5,0.8\},\{0.7,0.2\}\rangle\end{array}\right\}\\ r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.4,0.8,0.3\},\{0.6,0.8,0.6\},\{0.5,0.5\}\rangle,\\ \langle\pi_2,\{0.7,0.1\},\{0.7,0.4\},\{0.2,0.1,0.2\}\rangle,\\ \langle\pi_3,\{0.5,0.6,0.2,0.1\},\{0.4,0.7,0.3\},\{0.2,0.3,0.6\}\rangle,\\ \langle\pi_4,\{0.4,0.3\},\{0.4,0.5\},\{0.7,0.2,0.2\}\rangle\end{array}\right\}\end{array}\right\},$$

$$nhf(\widetilde{\delta}_1,\Re)\cap nhf(\widetilde{\delta}_2,\Re)$$
$$=\left\{\begin{array}{c} r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.2,0.7\},\{0.2,0.2,0\},\{0.6,0.6\}\rangle,\\ \langle\pi_2,\{0\},\{0,0\},\{1,1,1,1\}\rangle,\\ \langle\pi_3,\{0.3,0.1\},\{0.1\},\{0.7,0.8\}\rangle,\\ \langle\pi_4,\{0,0,0\},\{0,0\},\{1,1\}\rangle\end{array}\right\}\\ r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.2,0.7,0\},\{0.5,0.4,0.3\},\{0.8,0.5\}\rangle\\ \langle\pi_2,\{0.3,0\},\{0.3,0\},\{0.7,0.2,1\}\rangle\\ \langle\pi_3,\{0.3,0.1,0,0\},\{0.2,0.3,0\},\{0.4,0.5,0.7\}\rangle\\ \langle\pi_4,\{0.1,0.1\},\{0.2,0.4\},\{0.9,0.4,0.9\}\rangle\end{array}\right\}\end{array}\right\},$$

$$nhf(\widetilde{\delta}_1,\Re)\backslash nhf(\widetilde{\delta}_2,\Re)$$
$$=\left\{\begin{array}{c} r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.5,0\},\{0.7,0.2,0\},\{0.6,0.5,1\}\rangle,\\ \langle\pi_2,\{0\},\{0,0\},\{1,1,1,1\}\rangle,\\ \langle\pi_3,\{0.2,0.2\},\{0.3\},\{0.7,0.8\}\rangle,\\ \langle\pi_4,\{0,0\},\{0,0\},\{1,1,1\}\rangle\end{array}\right\}\\ r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.4,0.5\},\{0.5,0.6,0.6\},\{0.8,0.8,1\}\rangle\\ \langle\pi_2,\{0.2,0.1,0\},\{0.7,0\},\{0.7,1\}\rangle\\ \langle\pi_3,\{0.2,0.1,0\},\{0.2,0.3,0\},\{0.5,0.6,0.7,1\}\rangle\\ \langle\pi_4,\{0.4,0.1,0\},\{0.2,0.4\},\{0.9,0.4,1\}\rangle\end{array}\right\}\end{array}\right\},$$

$$nhf(\widetilde{\delta}_1,\Re)\wedge nhf(\widetilde{\delta}_2,\Re)$$
$$=\left\{\begin{array}{c} r_1\times r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.2,0.7\},\{0.2,0.2,0\},\{0.6,0.6\}\rangle,\\ \langle\pi_2,\{0\},\{0,0\},\{1,1,1,1\}\rangle,\\ \langle\pi_3,\{0.3,0.1\},\{0.1\},\{0.7,0.8\}\rangle,\\ \langle\pi_4,\{0,0,0\},\{0,0\},\{1,1\}\rangle\end{array}\right\}\\ r_1\times r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.2,0.5,0.3\},\{0.5,0.2,0.3\},\{0.6,0.5\}\rangle\\ \langle\pi_2,\{0.4\},\{0.3,0.4\},\{0.8,0.5,0.6,1\}\rangle\\ \langle\pi_3,\{0.3,0.2,0,0\},\{0.3,0,0\},\{0.7,0.8,1\}\rangle\\ \langle\pi_4,\{0,0\},\{0,0\},\{1,1,1\}\rangle\end{array}\right\}\\ r_2\times r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.2,0\},\{0.2,0.3,0\},\{0.8,0.6\}\rangle,\\ \langle\pi_2,\{0,0\},\{0\},\{1,1\}\rangle,\\ \langle\pi_3,\{0.3,0.1\},\{0.1,0\},\{0.4,0.4,1\}\rangle,\\ \langle\pi_4,\{0.4,0.1,0\},\{0.2,0.4\},\{0.9,0.4,1\}\rangle\end{array}\right\}\\ r_2\times r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.2,0.7,0\},\{0.5,0.4,0.3\},\{0.8,0.5\}\rangle\\ \langle\pi_2,\{0.3,0\},\{0.3,0\},\{0.7,0.2,1\}\rangle\\ \langle\pi_3,\{0.3,0.1,0,0\},\{0.2,0.3,0\},\{0.4,0.5,0.7\}\rangle\\ \langle\pi_4,\{0.1,0.1\},\{0.2,0.4\},\{0.9,0.4,0.9\}\rangle\end{array}\right\}\end{array}\right\},$$

$$nhf(\widetilde{\delta}_1,\Re)\vee nhf(\widetilde{\delta}_2,\Re)$$
$$=\left\{\begin{array}{c} r_1\times r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.5,0.7\},\{0.7,0.3,0.6\},\{0.4,0.5\}\rangle,\\ \langle\pi_2,\{0.4\},\{0.3,0.4\},\{0.8,0.5,0.6,0.2\}\rangle,\\ \langle\pi_3,\{0.5,0.2\},\{0.3\},\{0.2,0.4\}\rangle,\\ \langle\pi_4,\{0.7,0.4,0.6\},\{0.5,0.8\},\{0.7,0.2\}\rangle\end{array}\right\}\\ r_1\times r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.8,0.7\},\{0.7,0.4,0.6\},\{0.5,0.5\}\rangle\\ \langle\pi_2,\{0.7\},\{0.3,0.4\},\{0.2,0.1,0.2,0.2\}\rangle\\ \langle\pi_3,\{0.5,0.6,0.2,0.1\},\{0.4,0.7,0.3\},\{0.2,0.5,0.6\}\rangle\\ \langle\pi_4,\{0.1,0.3\},\{0.4,0.5\},\{0.7,0.2,0.9\}\rangle\end{array}\right\}\\ r_2\times r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.4,0.7,0.7\},\{0.6,0.8,0.6\},\{0.4,0.5\}\rangle,\\ \langle\pi_2,\{0.3,0.1\},\{0.7\},\{0.7,0.2\}\rangle,\\ \langle\pi_3,\{0.5,0.1\},\{0.2,0.3\},\{0.2,0.3,0.7\}\rangle,\\ \langle\pi_4,\{0.7,0.4,0.6\},\{0.5,0.8\},\{0.7,0.2,0.2\}\rangle\end{array}\right\}\\ r_2\times r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.4,0.8,0.3\},\{0.6,0.8,0.6\},\{0.5,0.5\}\rangle\\ \langle\pi_2,\{0.7,0.1\},\{0.7,0.4\},\{0.2,0.1,0.2\}\rangle\\ \langle\pi_3,\{0.5,0.6,0.2,0.1\},\{0.4,0.7,0.3\},\{0.2,0.3,0.6\}\rangle\\ \langle\pi_4,\{0.4,0.3\},\{0.4,0.5\},\{0.7,0.2,0.2\}\rangle\end{array}\right\}\end{array}\right\},$$

Definition 9 Let $\aleph=\{\pi_1,\pi_2,\ldots,\pi_n\}$ be a set of alternatives and $\Re=\{r_1,r_2,\ldots,r_m\}$ be a set of parameters. Let $nhf(\widetilde{\delta},\Re)$ be a neutrosophic hesitant fuzzy soft set over $NHFSS(\aleph,\Re)$

Let $\widetilde{t}(\pi),\widetilde{i}(\pi)\ \widetilde{f}(\pi)\subseteq[0,1]$ be hesitant subsets representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively. Then, neutrosophic hesitant fuzzy soft set $r_{\pi}^{t,i,f}$ is called a neutrosophic hesitant fuzzy soft point, for every $\pi\in\aleph,\ 0\leq\widetilde{t},\widetilde{i}\ \widetilde{f}\leq 1,\ r\in\Re$, and defined as follows.

$$r^{\pi}(\pi')(r')=\left\{\begin{array}{ll}(\{\widetilde{t}\},\{\widetilde{i}\},\{\widetilde{f}\}), & \text{if } r=r\prime \text{ and } \pi=\pi'\\ (\{0\},\{0\},\{1\}), & \text{otherwise}\end{array}\right.$$

Definition 10 Let $\aleph=\{\pi_1,\pi_2,\}$ and $\Re=\{r_1,r_2\}$. Then, a neutrosophic hesitant fuzzy soft sets $nhf(\widetilde{\delta}_1,\Re)$ can be expressed as follows:

$$nhf(\widetilde{\delta}_1,\Re)$$
$$=\left\{\begin{array}{c} r_1=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.5,0.7\},\{0.7,0.2,0.6\},\{0.6,0.5\}\rangle,\\ \langle\pi_2,\{0.4\},\{0.3,0.4\},\{0.8,0.5\}\rangle,\end{array}\right\}\\ r_2=\left\{\begin{array}{c}\langle\pi_1,\{0.4,0.7\},\{0.6,0.8,0.6\},\{0.8,0.5\}\rangle,\\ \langle\pi_2,\{0.3,0.1\},\{0.7\},\{0.7,0.2\}\rangle,\end{array}\right\}\end{array}\right\}$$

It is clear that $nhf(\widetilde{\delta}_1,\Re)$ is the union of its neutrosophic hesitant fuzzy soft points $r_{1_{\pi_1}}^{t,i,f},r_{1_{\pi_2}}^{t,i,f}\ r_{2_{\pi_1}}^{t,i,f},,r_{2_{\pi_2}}^{t,i,f}$. Some of these points are clearly written below.

$$r_{1_{\pi_1}}^{t,i,f}=\left\{\begin{array}{c}\langle\pi_1,\{0.3,0.5,0.7\},\{0.7,0.2,0.6\},\{0.6,0.5\}\rangle,\\ \langle\pi_2,\{0\},\{0\},\{1\}\rangle,\end{array}\right\},$$
$$r_{1_{\pi_2}}^{t,i,f}=\left\{\begin{array}{c}\langle\pi_1,\{0\},\{0\},\{1\}\rangle,\\ \langle\pi_2,\{0.4\},\{0.3,0.4\},\{0.8,0.5\}\rangle,\end{array}\right\},$$
$$r_{2_{\pi_1}}^{t,i,f}=\left\{\begin{array}{c}\langle\pi_1,\{0.4,0.7\},\{0.6,0.8,0.6\},\{0.8,0.5\}\rangle,\\ \langle\pi_2,\{0\},\{0\},\{1\}\rangle,\end{array}\right\},$$
$$r_{2_{\pi_2}}^{t,i,f}=\left\{\begin{array}{c}\langle\pi_1,\{0\},\{0\},\{1\}\rangle,\\ \langle\pi_2,\{0.3,0.1\},\{0.7\},\{0.7,0.2\}\rangle,\end{array}\right\}.$$

3.1 Geometric interpretation of NHFSS

To visualize the structure of a Neutrosophic Hesitant Fuzzy Soft Set, we consider the geometric representation within the Neutrosophic Cube.

Let $C=[0,1]^3$ be the unit neutrosophic cube where the three orthogonal axes represent Truth (T), Indeterminacy (I), and Falsity (F). For a fixed parameter $r\in\Re$ and a fixed element $\pi\in\aleph$, a Single-Valued Neutrosophic Soft evaluation corresponds to a single point $P(t,i,f)$ inside this cube.

However, in the proposed NHFSS framework, the evaluation is not a single point but a set of hesitant values for each dimension. Geometrically, an NHFSS element

$$\Phi(r,\pi)=\langle\tilde{t}(\pi),\tilde{i}(\pi),\tilde{f}(\pi)\rangle$$

defines a **rectangular prism (or a hyper-region)** within the neutrosophic cube bounded by the minimum and maximum values of the hesitant sets.

Let $t^-=\min(\tilde{t})$, $t^+=\max(\tilde{t})$, $i^-=\min(\tilde{i})$, $i^+=\max(\tilde{i})$, and $f^-=\min(\tilde{f})$, $f^+=\max(\tilde{f})$. The geometric existence of the NHFSS element for a specific pair (r,π) is the region $\mathcal{R}$ defined by:

$$\mathcal{R}_{(r,\pi)}=[t^-,t^+]\times[i^-,i^+]\times[f^-,f^+]\subset[0,1]^3 \quad (1)$$

This geometric region represents the "space of hesitation" for the decision maker. A larger volume indicates higher hesitation and uncertainty, whereas a single point (volume = 0) indicates precise neutrosophic knowledge (Fig. 1).

4 Neutrosophic hesitant fuzzy soft topological spaces

Definition 11 Let $NHFSS(\aleph, \Re)$ be the family of all neutrosophic hesitant fuzzy soft sets over the universe set $\aleph$ and $\tilde{\tau}^{nhf} \subset NHFSS(\aleph, \Re)$. Then $\tilde{\tau}^{nhf}$ is said to be a neutrosophic hesitant fuzzy soft topology on $\aleph$ if

1. $nhf(\tilde{0}_{(\aleph,\Re)}) \in \tilde{\tau}^{nhf}$ and $nhf(\widetilde{1}_{(\aleph,\Re)}) \in \tilde{\tau}^{nhf}$,
2. Arbitrary union of members in $\tilde{\tau}^{nhf}$ is also in $\tilde{\tau}^{nhf}$,
3. Finite intersection of members in $\tilde{\tau}^{nhf}$ is also in $\tilde{\tau}^{nhf}$

Then $(\aleph, \tilde{\tau}^{nhf}, \Re)$ is said to be a neutrosophic hesitant fuzzy soft topological space over $\aleph$. Each members of $\tilde{\tau}^{nhf}$ is said to be neutrosophic hesitant fuzzy soft open set.

Definition 12 Let $(\aleph, \tilde{\tau}^{nhf}, \Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$ and $nhf(\widetilde{\delta}, \Re)$ be a neutrosophic hesitant fuzzy soft set over $\aleph$. Then $nhf(\widetilde{\delta}, \Re)$ is said to be neutrosophic hesitant fuzzy soft closed set iff its complement is a neutrosophic hesitant fuzzy soft open set.

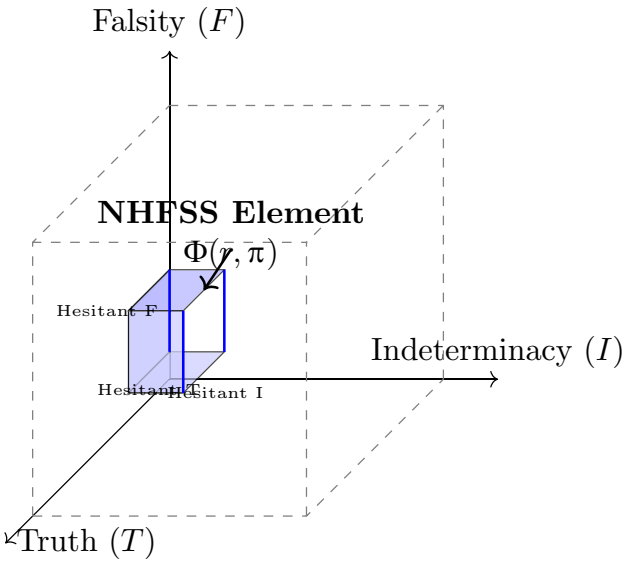

Fig. 1 Geometric representation of an NHFSS element as a hesitant region within the Neutrosophic Cube

Example 3 Let $\aleph = \{\pi_1, \pi_2, \}$ and $\Re = \{r_1, r_2\}$. Let us consider

$$nhf(\widetilde{\delta}_1, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.1, 0.4\}, \{0.3\}, \{0.4, 0.2, 0.5\} \rangle, \\ \langle \pi_2, \{0.5\}, \{0.2, 0.6\}, \{0.4, 0.1\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.3\}, \{0.4, 0.2\}, \{0.3, 0.2, 0.1\} \rangle, \\ \langle \pi_2, \{0.3, 0.2\}, \{0.3, 0.6\}, \{0.4, 0.1\} \rangle, \end{array} \right\} \end{array} \right\},$$

$$nhf(\widetilde{\delta}_2, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.5, 0.3, 0.1\}, \{0.2, 0.8\}, \{0.4, 0.7\} \rangle, \\ \langle \pi_2, \{0.2, 0.4\}, \{0.2\}, \{0.4, 0.3, 0.7\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.2, 0.3\}, \{0.7\}, \{0.8\} \rangle, \\ \langle \pi_2, \{0.7, 0.6\}, \{0.2\}, \{0.3, 0.4\} \rangle, \end{array} \right\} \end{array} \right\},$$

$$nhf(\widetilde{\delta}_3, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.5, 0.4, 0.1\}, \{0.3, 0.8\}, \{0.4, 0.2, 0.5\} \rangle, \\ \langle \pi_2, \{0.5, 0.4\}, \{0.2, 0.6\}, \{0.4, 0.1, 0.7\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.3, 0.3\}, \{0.7, 0.2\}, \{0.3, 0.2, 0.1\} \rangle, \\ \langle \pi_2, \{0.7, 0.6\}, \{0.3, 0.6\}, \{0.3, 0.1\} \rangle, \end{array} \right\} \end{array} \right\},$$

$$nhf(\widetilde{\delta}_4, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.1, 0.3\}, \{0.2\}, \{0.4, 0.7\} \rangle, \\ \langle \pi_2, \{0.2\}, \{0.2\}, \{0.4, 0.3\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.2\}, \{0.4\}, \{0.8\} \rangle, \\ \langle \pi_2, \{0.3, 0.2\}, \{0.2\}, \{0.4, 0.4\} \rangle, \end{array} \right\} \end{array} \right\},$$

any neutrosophic hesitant fuzzy soft sets in $\aleph$. We see that

$$\tilde{\tau}^{nhf} = \{nhf(\tilde{0}_{(\aleph,\Re)}), nhf(\widetilde{1}_{(\aleph,\Re)}), nhf(\widetilde{\delta}_1, \Re), nhf(\widetilde{\delta}_2, \Re), nhf\left(\widetilde{\delta}_3, \Re\right), nhf(\widetilde{\delta}_4, \Re)\}$$

is neutrosophic hesitant fuzzy soft topology in $\aleph$.

Proposition 5 *Let $(\aleph, \tilde{\tau}^{nhf}, \Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$. Then*

1. *$nhf(\tilde{0}_{(\aleph,\Re)})$ and $nhf(\widetilde{1}_{(\aleph,\Re)})$ are neutrosophic hesitant fuzzy soft closed sets over $\aleph$,*
2. *$(\bigcap_{i\in I} nhf(\widetilde{\delta}_i, \Re))$ is a neutrosophic hesitant fuzzy soft closed set over $\aleph$, where each $nhf(\widetilde{\delta}_i, \Re)$ is an neutrosophic closed set;*
3. *$(\bigcup_{i=1}^{n} nhf(\widetilde{\delta}_i, \Re))$ is a neutrosophic hesitant fuzzy soft closed set over $\aleph$, where each $nhf(\widetilde{\delta}_i, \Re)$ is an neutrosophic closed set;.*

Definition 13 Let $NHFSS(\aleph, \Re)$ be the family of all neutrosophic hesitant fuzzy soft sets over the universe set $\aleph$.

1. If $\tilde{\tau}^{nhf} = \{nhf(\tilde{0}_{(\aleph,\Re)}),\ nhf(\widetilde{1}_{(\aleph,\Re)})\}$, then $\tilde{\tau}^{nhf}$ is said to be the neutrosophic hesitant fuzzy soft indiscrete topology and $(\aleph, \tilde{\tau}^{nhf}, \Re)$ is said to be a neutrosophic hesitant fuzzy soft indiscrete topological space over $\aleph$.
2. If $\tilde{\tau}^{nhf} = NHFSS(\aleph, \Re)$, then $\tilde{\tau}^{nhf}$ is said to be the neutrosophic hesitant fuzzy soft discrete topology and

$(\aleph, \tilde{\tau}^{nhf}, \Re)$ is said to be a neutrosophic hesitant fuzzy soft discrete topological space over $\aleph$.

Proposition 6 *Let* $(\aleph, {}^{NSS}\tau_1, \Re)$ *and* $(\aleph, {}^{NSS}\tau_2, \Re)$ *be two neutrosophic hesitant fuzzy soft topological spaces over the same universe set* $\aleph$. *Then* $(\aleph, {}^{NSS}\tau_1 \cap {}^{NSS}\tau_2, \Re)$ *is neutrosophic hesitant fuzzy soft topological space over* $\aleph$.

Proof 1. Since $nhf(\tilde{0}_{(\aleph,\Re)})$, $nhf(\widetilde{1}_{(\aleph,\Re)}) \in {}^{NSS}\tau_1$ and $nhf(\tilde{0}_{(\aleph,\Re)})$, $nhf(\widetilde{1}_{(\aleph,\Re)}) \in {}^{NSS}\tau_2$, then $nhf(\tilde{0}_{(\aleph,\Re)})$, $nhf(\widetilde{1}_{(\aleph,\Re)}) \in {}^{NSS}\tau_1 \cap {}^{NSS}\tau_2$.

2. Suppose that $\{(\widetilde{\delta}_i, \Re) | i \in \widetilde{i}\}$ be a family of neutrosophic hesitant fuzzy soft sets in ${}^{NSS}\tau_1 \cap {}^{NSS}\tau_2$. Then $(\widetilde{\delta}_i, \Re) \in {}^{NSS}\tau_1$ and $(\widetilde{\delta}_i, \Re) \in {}^{NSS}\tau_2$ for all $i \in \widetilde{i}$, so $\cup_{i\in\widetilde{i}}(\widetilde{\delta}_i, \Re) \in \tau_1^{NSS}$ and $\cup_{i\in\widetilde{i}}(\widetilde{\delta}_i, \Re) \in \tau_2^{NSS}$. Thus $\cup_{i\in\widetilde{i}}(\widetilde{\delta}_i, \Re) \in \tau_1^{NSS} \cap \tau_2^{NSS}$.

3. Let $\{(\widetilde{\delta}_i, \Re) | i = \overline{1,n}\}$ be a family of the finite number of neutrosophic hesitant fuzzy soft sets in ${}^{NSS}\tau_1 \; {}^{NSS}\tau_2$. Then $(\widetilde{\delta}_i, \Re) \; {}^{NSS}\tau_1$ and $(\widetilde{\delta}_i, \Re) \in {}^{NSS}\tau_2$ for $i = \overline{1,n}$, so $\cap_{i=1}^{n}(\widetilde{\delta}_i, \Re) \in \tau_1^{NSS}$ and $\cap_{i=1}^{n}(\widetilde{\delta}_i, \Re) \in \tau_2^{NSS}$. Thus $\cap_{i=1}^{n}(\widetilde{\delta}_i, \Re) \in \tau_1^{NSS} \cap \tau_2^{NSS}$. □

Remark 2 The union of two neutrosophic hesitant fuzzy soft topologies over $\aleph$ may not be a neutrosophic hesitant fuzzy soft topology on $\aleph$.

Example 4 Let $\aleph = \{\pi_1, \pi_2\}$ be an initial universe set, $\Re = \{r_1, r_2\}$ be a set of parameters and ${}^{NSS}\tau_1 = \{nhf(\tilde{0}_{(\aleph,\Re)}), nhf(\widetilde{1}_{(\aleph,\Re)}), nhf(\widetilde{\delta}_1, \Re), nhf(\widetilde{\delta}_2, \Re), nhf(\widetilde{\delta}_3, \Re)\}$ and ${}^{NSS}\tau_2 = \{nhf(\tilde{0}_{(\aleph,\Re)}), \; nhf(\widetilde{1}_{(\aleph,\Re)}), \; nhf(\widetilde{\delta}_2, \Re), \; nhf(\widetilde{\delta}_4, \Re)\}$ be two neutrosophic hesitant fuzzy soft topologies over $\aleph$. Here, the neutrosophic hesitant fuzzy soft sets $nhf(\widetilde{\delta}_1, \Re)$, $nhf(\widetilde{\delta}_2, \Re)$ $nhf(\widetilde{\delta}_3, \Re)$ and $nhf(\widetilde{\delta}_4, \Re)$ over $\aleph$ are defined as following: Let $\aleph = \{\pi_1, \pi_2, \}$ and $\Re = \{r_1, r_2\}$. Let us consider

$$nhf(\widetilde{\delta}_1, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.5, 0.4, 0.1\}, \{0.3, 0.8\}, \{0.4, 0.2, 0.5\} \rangle, \\ \langle \pi_2, \{0.5, 0.4\}, \{0.2, 0.6\}, \{0.4, 0.1, 0.7\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.3, 0.3\}, \{0.7, 0.2\}, \{0.3, 0.2, 0.1\} \rangle, \\ \langle \pi_2, \{0.7, 0.6\}, \{0.3, 0.6\}, \{0.3, 0.1\} \rangle, \end{array} \right\} \end{array} \right\},$$

$$nhf(\widetilde{\delta}_2, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.1, 0.3\}, \{0.2\}, \{0.4, 0.7, 0.7\} \rangle, \\ \langle \pi_2, \{0.2\}, \{0.2\}, \{0.4, 0.3, 0.8\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.2\}, \{0.4\}, \{0.8, 0.3, 0.4\} \rangle, \\ \langle \pi_2, \{0.3, 0.2\}, \{0.2\}, \{0.5, 0.4\} \rangle, \end{array} \right\} \end{array} \right\},$$

$$nhf(\widetilde{\delta}_3, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.3, 0.4\}, \{0.3\}, \{0.4, 0.3, 0.6\} \rangle, \\ \langle \pi_2, \{0.4\}, \{0.2, 0.5\}, \{0.4, 0.1, 0.7\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.2\}, \{0.6, 0.1\}, \{0.4, 0.2, 0.1\} \rangle, \\ \langle \pi_2, \{0.6, 0.4\}, \{0.3, 0.3\}, \{0.4, 0.1\} \rangle, \end{array} \right\} \end{array} \right\},$$

$$nhf(\widetilde{\delta}_4, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.6, 0.3\}, \{0.6, 0.2\}, \{0.3, 0.6, 0.5\} \rangle, \\ \langle \pi_2, \{0.4, 0.3\}, \{0.6, 0.4\}, \{0.3, 0.2, 0.7\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.5, 0.2\}, \{0.8, 0.3\}, \{0.4, 0.2, 0.3\} \rangle, \\ \langle \pi_2, \{0.4, 0.4\}, \{0.5\}, \{0.2, 0.3\} \rangle, \end{array} \right\} \end{array} \right\},$$

Since $nhf(\widetilde{\delta}_1, \Re) \cup nhf(\widetilde{\delta}_4, \Re) \notin {}^{NSS}\tau_1 \cup {}^{NSS}\tau_2$, then ${}^{NSS}\tau_1 \cup {}^{NSS}\tau_2$ is not a neutrosophic hesitant fuzzy soft topology over $\aleph$.

Definition 14 Let $(\aleph, \tilde{\tau}^{nhf}, \Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$ and $nhf(\widetilde{\delta}, \Re) \in NHFSS(\aleph, \Re)$ be a neutrosophic hesitant fuzzy soft set. Then, the neutrosophic hesitant fuzzy soft interior of $nhf(\widetilde{\delta}, \Re)$, denoted $nhf(\widetilde{\delta}, \Re)^{\circ}$, is defined as the neutrosophic hesitant fuzzy soft union of all neutrosophic hesitant fuzzy soft open subsets of $nhf(\widetilde{\delta}, \Re)$.

Clearly, $nhf(\widetilde{\delta}, \Re)^{\circ}$ is the biggest neutrosophic hesitant fuzzy soft open set that is contained by $nhf(\widetilde{\delta}, \Re)$.

Example 5 Let us consider the neutrosophic hesitant fuzzy soft topology ${}^{NSS}\tau_1$ given in Example-4. Suppose that an any $nhf(\widetilde{\delta}, \Re) \in NHFSS(\aleph, \Re)$ is defined as following:

$$nhf(\widetilde{\delta}, \Re) = \left\{ \begin{array}{c} r_1 = \left\{ \begin{array}{c} \langle \pi_1, \{0.4, 0.6\}, \{0.5\}, \{0.2, 0.1, 0.2\} \rangle, \\ \langle \pi_2, \{0.5\}, \{0.4, 0.5\}, \{0.2, 0.1, 0.4\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{c} \langle \pi_1, \{0.3, 0.3\}, \{0.7, 0.3\}, \{0.1, 0.1, 0.1\} \rangle, \\ \langle \pi_2, \{0.8, 0.6\}, \{0.4, 0.5\}, \{0.1, 0.1\} \rangle, \end{array} \right\} \end{array} \right\},$$

Then $nhf(\tilde{0}_{(\aleph,\Re)})$, $nhf(\widetilde{\delta}_2, \Re)$, $nhf(\widetilde{\delta}_3, \Re) \subseteq nhf(\widetilde{\delta}, \Re)$. Therefore, $nhf(\widetilde{\delta}, \Re)^{\circ} = nhf(\tilde{0}_{(\aleph,\Re)}) \cup nhf(\widetilde{\delta}_2, \Re) \cup nhf(\widetilde{\delta}_3, \Re) = nhf(\widetilde{\delta}_3, \Re)$.

Theorem 7 Let $(\aleph, \tilde{\tau}^{nhf}, \Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$ and $nhf(\widetilde{\delta}, \Re) \in NHFSS(\aleph, \Re)$. $nhf(\widetilde{\delta}, \Re)$ is a neutrosophic hesitant fuzzy soft open set iff $nhf(\widetilde{\delta}, \Re)$ $nhf(\widetilde{\delta}, \Re)^{\circ}$. □

Proof Let $nhf(\widetilde{\delta}, \Re)$ be a neutrosophic hesitant fuzzy soft open set. Then the biggest neutrosophic hesitant fuzzy soft open set that is contained by $nhf(\widetilde{\delta}, \Re)$ is equal to $nhf(\widetilde{\delta}, \Re)$. Hence, $nhf(\widetilde{\delta}, \Re) = nhf(\widetilde{\delta}, \Re)^{\circ}$.

Conversely, it is known that $nhf(\widetilde{\delta}, \Re)^{\circ}$ is a neutrosophic hesitant fuzzy soft open set and if $nhf(\widetilde{\delta}, \Re) = nhf(\widetilde{\delta}, \Re)^{\circ}$, then $nhf(\widetilde{\delta}, \Re)$ is a neutrosophic hesitant fuzzy soft open set. □

Theorem 8 Let $(\aleph, \tilde{\tau}^{nhf}, \Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$ and $nhf(\widetilde{\delta}_1, \Re)$, $nhf(\widetilde{\delta}_2, \Re) \in NHFSS(\aleph, \Re)$. Then,

1. $[nhf(\widetilde{\delta}_1, \Re)^{\circ}]^{\circ} = nhf(\widetilde{\delta}_1, \Re)^{\circ}$,

2. $(nhf(\tilde{0}_{(\aleph,\Re)}))^\circ = nhf(\tilde{0}_{(\aleph,\Re)})$ and $(nhf(\tilde{1}_{(\aleph,\Re)}))^\circ = nhf(\tilde{1}_{(\aleph,\Re)})$,
3. $nhf(\tilde{\delta}_1,\Re) \subseteq nhf(\tilde{\delta}_2,\Re) \Rightarrow nhf(\tilde{\delta}_1,\Re)^\circ \subseteq nhf(\tilde{\delta}_2,\Re)^\circ$,
4. $[nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]^\circ = nhf(\tilde{\delta}_1,\Re)^\circ \cap nhf(\tilde{\delta}_2,\Re)^\circ$,
5. $nhf(\tilde{\delta}_1,\Re)^\circ \cup nhf(\tilde{\delta}_2,\Re)^\circ \subseteq [nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]^\circ$.

Proof 1. Let $nhf(\tilde{\delta}_1,\Re)^\circ nhf(\tilde{\delta}_2,\Re)$. Then $nhf(\tilde{\delta}_2,\Re) \in \tilde{\tau}^{nhf}$ iff $nhf(\tilde{\delta}_2,\Re) = nhf(\tilde{\delta}_2,\Re)^\circ$. So, $[nhf(\tilde{\delta}_1,\Re)^\circ]^\circ = nhf(\tilde{\delta}_1,\Re)^\circ$.

2. Straighforward.

3. It is known that $nhf(\tilde{\delta}_1,\Re)^\circ \subseteq nhf(\tilde{\delta}_1,\Re) \subseteq nhf(\tilde{\delta}_2,\Re)$ and $nhf(\tilde{\delta}_2,\Re)^\circ \subseteq nhf(\tilde{\delta}_2,\Re)$. Since $nhf(\tilde{\delta}_2,\Re)^\circ$ is the biggest neutrosophic hesitant fuzzy soft open set contained in $nhf(\tilde{\delta}_2,\Re)$ and so, $nhf(\tilde{\delta}_1,\Re)^\circ \subseteq nhf(\tilde{\delta}_2,\Re)^\circ$.

4. Since $nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re) \subseteq nhf(\tilde{\delta}_1,\Re)$ and $nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re) \subseteq nhf(\tilde{\delta}_2,\Re)$, then $[nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]^\circ \subseteq nhf(\tilde{\delta}_1,\Re)^\circ$ and $[nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]^\circ \subseteq nhf(\tilde{\delta}_2,\Re)^\circ$ and so, $[nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]^\circ \subseteq nhf(\tilde{\delta}_1,\Re)^\circ \cap nhf(\tilde{\delta}_2,\Re)^\circ$.

On the other hand, since $nhf(\tilde{\delta}_1,\Re)^\circ \subseteq nhf(\tilde{\delta}_1,\Re)$ and $nhf(\tilde{\delta}_2,\Re)^\circ \subseteq nhf(\tilde{\delta}_2,\Re)$, then $nhf(\tilde{\delta}_1,\Re)^\circ \cap nhf(\tilde{\delta}_2,\Re)^\circ \subseteq nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)$. Besides, $[nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]^\circ \subseteq nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)$ and it is the biggest neutrosophic hesitant fuzzy soft open set. Therefore, $nhf(\tilde{\delta}_1,\Re)^\circ \cap nhf(\tilde{\delta}_2,\Re)^\circ \subseteq [nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]^\circ$. Thus, $[nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]^\circ = nhf(\tilde{\delta}_1,\Re)^\circ \cap nhf(\tilde{\delta}_2,\Re)^\circ$.

5. Since $nhf(\tilde{\delta}_1,\Re) \subseteq nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)$ and $nhf(\tilde{\delta}_2,\Re) \subseteq nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)$, then $nhf(\tilde{\delta}_1,\Re)^\circ \subseteq [nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]^\circ$ and $nhf(\tilde{\delta}_2,\Re)^\circ \subseteq [nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]^\circ$. Therefore, $nhf(\tilde{\delta}_1,\Re)^\circ \cup nhf(\tilde{\delta}_2,\Re)^\circ \subseteq [nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]^\circ$. □

Definition 15 Let $(\aleph,\tilde{\tau}^{nhf},\Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$ and $nhf(\tilde{\delta},\Re) \in NHFSS(\aleph,\Re)$ be a neutrosophic hesitant fuzzy soft set. Then, the neutrosophic hesitant fuzzy soft closure of $nhf(\tilde{\delta},\Re)$, denoted $\overline{nhf(\tilde{\delta},\Re)}$, is defined as the neutrosophic hesitant fuzzy soft intersection of all neutrosophic hesitant fuzzy soft closed supersets of $nhf(\tilde{\delta},\Re)$.

Clearly, $\overline{nhf(\tilde{\delta},\Re)}$ is the smallest neutrosophic hesitant fuzzy soft closed set that containing $nhf(\tilde{\delta},\Re)$.

Example 6 Let us consider the neutrosophic hesitant fuzzy soft topology ${}^{NSS}\tau_1$ given in Example-4. Suppose that an any $nhf(\tilde{\delta},\Re) \in NHFSS(\aleph,\Re)$ is defined as following:

$$nhf(\tilde{\delta},\Re) = \left\{ \begin{array}{l} r_1 = \left\{ \begin{array}{l} \langle \pi_1, \{0.2,0.1,0.3\}, \{0.5,0.1\}, \{0.7,0.6,0.3\} \rangle, \\ \langle \pi_2, \{0.3,0.1,0.4\}, \{0.7,0.2\}, \{0.6,0.7\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{l} \langle \pi_1, \{0.2,0.1,0.1\}, \{0.2,0.5\}, \{0.5,0.7\} \rangle, \\ \langle \pi_2, \{0.2,0.1\}, \{0.3,0.2\}, \{0.8,0.7\} \rangle, \end{array} \right\} \end{array} \right\},$$

Obviously, $nhf(\tilde{0}_{(\aleph,\Re)})$, $nhf(\tilde{1}_{(\aleph,\Re)})$, $nhf(\tilde{\delta}_1,\Re)^c$, $nhf(\tilde{\delta}_2,\Re)^c$ and $nhf(\tilde{\delta}_3,\Re)^c$ are all neutrosophic hesitant fuzzy soft closed sets over $(\aleph, {}^{NSS}\tau_1, \Re)$. They are given as following:

$$nhf(\tilde{0}_{(\aleph,\Re)}) = nhf\left(\tilde{1}_{(\aleph,\Re)}\right),$$

$$nhf(\tilde{1}_{(\aleph,\Re)}) = nhf(\tilde{0}_{(\aleph,\Re)})$$

$$nhf(\tilde{\delta}_1,\Re)^c = \left\{ \begin{array}{l} r_1 = \left\{ \begin{array}{l} \langle \pi_1, \{0.4,0.2,0.5\}, \{0.7,0.2\}, \{0.5,0.4,0.1\} \rangle, \\ \langle \pi_2, \{0.4,0.1,0.7\}, \{0.8,0.4\}, \{0.5,0.4\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{l} \langle \pi_1, \{0.3,0.2,0.1\}, \{0.3,0.8\}, \{0.3,0.3\} \rangle, \\ \langle \pi_2, \{0.3,0.1\}, \{0.7,0.4\}, \{0.7,0.6\} \rangle, \end{array} \right\} \end{array} \right\},$$

$$nhf(\tilde{\delta}_2,\Re)^c = \left\{ \begin{array}{l} r_1 = \left\{ \begin{array}{l} \langle \pi_1, \{0.4,0.7,0.7\}, \{0.8,1\}, \{0.1,0.3,0\} \rangle, \\ \langle \pi_2, \{0.4,0.3,0.8\}, \{0.8,1\}, \{0.2,0\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{l} \langle \pi_1, \{0.8,0.3,0.4\}, \{0.6,1\}, \{0.2,0\} \rangle, \\ \langle \pi_2, \{0.5,0.4\}, \{0.8,1\}, \{0.3,0.2\} \rangle, \end{array} \right\} \end{array} \right\},$$

$$nhf(\tilde{\delta}_3,\Re) = \left\{ \begin{array}{l} r_1 = \left\{ \begin{array}{l} \langle \pi_1, \{0.4,0.3,0.6\}, \{0.7,1\}, \{0.3,0.4,0\} \rangle, \\ \langle \pi_2, \{0.4,0.1,0.7\}, \{0.8,0.5\}, \{0.4,0\} \rangle, \end{array} \right\} \\ r_2 = \left\{ \begin{array}{l} \langle \pi_1, \{0.4,0.2,0.1\}, \{0.4,0.9\}, \{0.2,0\} \rangle, \\ \langle \pi_2, \{0.4,0.1\}, \{0.7,0.7\}, \{0.6,0.4\} \rangle, \end{array} \right\} \end{array} \right\},$$

Then $nhf(\tilde{\delta},\Re) \subset nhf(\tilde{1}_{(\aleph,\Re)})$, $\overline{nhf(\tilde{\delta}_1,\Re)^c}$, $nhf(\tilde{\delta}_2,\Re)^c$, $nhf(\tilde{\delta}_3,\Re)^c$. Therefore, $\overline{nhf(\tilde{\delta},\Re)} = nhf(\tilde{1}_{(\aleph,\Re)}) \cap nhf(\tilde{\delta}_1,\Re)^c \cap nhf(\tilde{\delta}_2,\Re)^c \cap nhf(\tilde{\delta}_3,\Re)^c = nhf(\tilde{\delta}_1,\Re)^c$.

Theorem 9 Let $(\aleph,\tilde{\tau}^{nhf},\Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$ and $nhf(\tilde{\delta},\Re) \in NHFSS(\aleph,\Re)$. $nhf(\tilde{\delta},\Re)$ is neutrosophic hesitant fuzzy soft closed set iff $nhf(\tilde{\delta},\Re) = \overline{nhf(\tilde{\delta},\Re)}$.

Proof Straightforward. □

Theorem 10 Let $(\aleph,\tilde{\tau}^{nhf},\Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$ and $nhf(\tilde{\delta}_1,\Re)$, $nhf(\tilde{\delta}_2,\Re) \in NHFSS(\aleph,\Re)$. Then,

1. $\overline{\overline{[nhf(\tilde{\delta}_1,\Re)]}} = \overline{nhf(\tilde{\delta}_1,\Re)}$,

2. $\overline{(nhf(\tilde{0}_{(\aleph,\Re)}))} = nhf(\tilde{0}_{(\aleph,\Re)})$ and $\overline{(nhf(\tilde{1}_{(\aleph,\Re)}))} = nhf(\tilde{1}_{(\aleph,\Re)})$
3. $nhf(\tilde{\delta}_1,\Re) \subseteq nhf(\tilde{\delta}_2,\Re) \Rightarrow \overline{nhf(\tilde{\delta}_1,\Re)} \subseteq \overline{nhf(\tilde{\delta}_2,\Re)}$,
4. $\overline{[nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]} \Rightarrow \overline{nhf(\tilde{\delta}_1,\Re)} \cup \overline{nhf(\tilde{\delta}_2,\Re)}$,
5. $\overline{[nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]}\ \overline{nhf(\tilde{\delta}_1,\Re)} \cap \overline{nhf(\tilde{\delta}_2,\Re)}$.

Proof 1. Let $\overline{nhf(\tilde{\delta}_1,\Re)} = nhf(\tilde{\delta}_2,\Re)$. Then, $nhf(\tilde{\delta}_2,\Re)$ is a neutrosophic hesitant fuzzy soft closed set. Hence, $\overline{nhf(\tilde{\delta}_2,\Re)}$ and $\overline{nhf(\tilde{\delta}_2,\Re)}$ are equal. Therefore, $\overline{[nhf(\tilde{\delta}_1,\Re)]} = \overline{nhf(\tilde{\delta}_1,\Re)}$.

2. Straightforward.

3. It is known that $nhf(\tilde{\delta}_1,\Re) \subseteq \overline{nhf(\tilde{\delta}_1,\Re)}$ and $nhf(\tilde{\delta}_2,\Re) \subseteq \overline{nhf(\tilde{\delta}_2,\Re)}$ and so, $nhf(\tilde{\delta}_1,\Re) \subseteq nhf(\tilde{\delta}_2,\Re) \subseteq \overline{nhf(\tilde{\delta}_2,\Re)}$. Since $\overline{nhf(\tilde{\delta}_1,\Re)}$ is the smallest neutrosophic hesitant fuzzy soft closed set containing $nhf(\tilde{\delta}_1,\Re)$, then $\overline{nhf(\tilde{\delta}_1,\Re)} \cup \subseteq \overline{nhf(\tilde{\delta}_2,\Re)}$.

4. Since $nhf(\tilde{\delta}_1,\Re) \subseteq nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)$ and $nhf(\tilde{\delta}_2,\Re) \subseteq nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)$, then $\overline{nhf(\tilde{\delta}_1,\Re)} \subseteq \overline{[nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]}$ and $\overline{nhf(\tilde{\delta}_2,\Re)} \subseteq \overline{[nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]}$ and so, $\overline{nhf(\tilde{\delta}_1,\Re)} \cup \overline{nhf(\tilde{\delta}_2,\Re)} \subseteq \overline{[nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)}$.

Conversely, since $nhf(\tilde{\delta}_1,\Re) \subseteq \overline{nhf(\tilde{\delta}_1,\Re)}$ and $nhf(\tilde{\delta}_2,\Re) \subseteq \overline{nhf(\tilde{\delta}_2,\Re)}$, then $nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re) \subseteq \overline{nhf(\tilde{\delta}_1,\Re)} \cup \overline{nhf(\tilde{\delta}_2,\Re)}$. Besides, $\overline{[nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]}$ is the smallest neutrosophic hesitant fuzzy soft closed set that containing $nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)$. Therefore, $\overline{[nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]} \subseteq \overline{nhf(\tilde{\delta}_1,\Re)} \cup \overline{nhf(\tilde{\delta}_2,\Re)}]$. Thus, $\overline{[nhf(\tilde{\delta}_1,\Re) \cup nhf(\tilde{\delta}_2,\Re)]} = \overline{nhf(\tilde{\delta}_1,\Re)} \cup \overline{nhf(\tilde{\delta}_2,\Re)}$.

5. Since $nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re) \subseteq \overline{nhf(\tilde{\delta}_1,\Re)} \cap \overline{nhf(\tilde{\delta}_2,\Re)}$ and $\overline{[nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)]}$ is the smallest neutrosophic hesitant fuzzy soft closed set that containing $nhf(\tilde{\delta}_1,\Re) \cap nhf(\tilde{\delta}_2,\Re)$, then $\overline{[nhf(\tilde{\delta}_1,\Re)} \cap \overline{nhf(\tilde{\delta}_2,\Re)} \subseteq \overline{nhf(\tilde{\delta}_1,\Re)} \cap \overline{nhf(\tilde{\delta}_2,\Re)}$. □

Theorem 11 Let $(\aleph, \tilde{\tau}^{nhf}, \Re)$ be a neutrosophic hesitant fuzzy soft topological space over $\aleph$ and $nhf(\tilde{\delta},\Re) \in NHFSS(\aleph,\Re)$. Then,

1. $[\overline{nhf(\tilde{\delta},\Re)}]^c = [nhf(\tilde{\delta},\Re)^c]^\circ$,
2. $[nhf(\tilde{\delta},\Re)^\circ]^c = \overline{[nhf(\tilde{\delta},\Re)^c]}$.

Proof 1. $\overline{nhf(\tilde{\delta},\Re)} = \cap\{(\tilde{G},\Re) \in \overset{NSS}{\tau^c} : (\tilde{G},\Re) \supseteq nhf(\tilde{\delta},\Re)\}$

$\Longrightarrow [\overline{nhf(\tilde{\delta},\Re)}]^c = [\cap\{(\tilde{G},\Re) \in \overset{NSS}{\tau^c} : (\tilde{G},\Re) \supseteq nhf(\tilde{\delta},\Re)\}]^c = \cup\{(\tilde{G},\Re)^c \in \tilde{\tau}^{nhf} : (\tilde{G},\Re)^c \subseteq nhf(\tilde{\delta},\Re)^c\} = [nhf(\tilde{\delta},\Re)^c]^\circ$.

2. $nhf(\tilde{\delta},\Re)^\circ = \cup\{(\tilde{G},\Re) \in \tilde{\tau}^{nhf} : (\tilde{G},\Re) \subseteq nhf(\tilde{\delta},\Re)\}$

$\Longrightarrow [nhf(\tilde{\delta},\Re)^\circ]^c = [\cup\{(\tilde{G},\Re) \in \tilde{\tau}^{nhf} : (\tilde{G},\Re) \subseteq nhf(\tilde{\delta},\Re)\}]^c = \cap\{(\tilde{G},\Re)^c \in \overset{NSS}{\tau^c} : (\tilde{G},\Re)^c \supseteq nhf(\tilde{\delta},\Re)^c\} = \overline{[nhf(\tilde{\delta},\Re)^c]}$. □

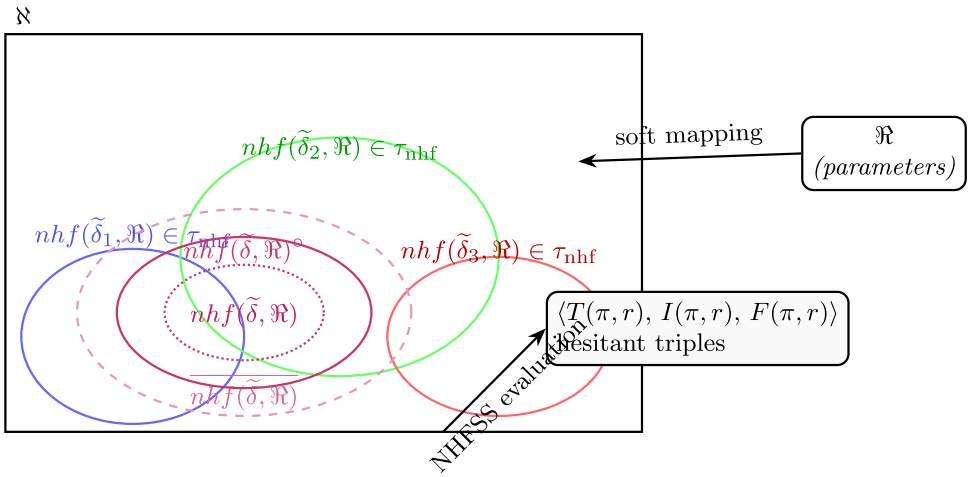

Fig. 2 A schematic representation of a neutrosophic hesitant fuzzy soft topological space $(\aleph, \tau_{\text{nhf}}, \Re)$, showing some $nhf(\tilde{\delta}_i,\Re) \in \tau_{\text{nhf}}$ as open sets, together with the interior $nhf(\tilde{\delta},\Re)^\circ$ and closure $\overline{nhf(\tilde{\delta},\Re)}$ of a specific NHFSS-set under the soft parameter structure

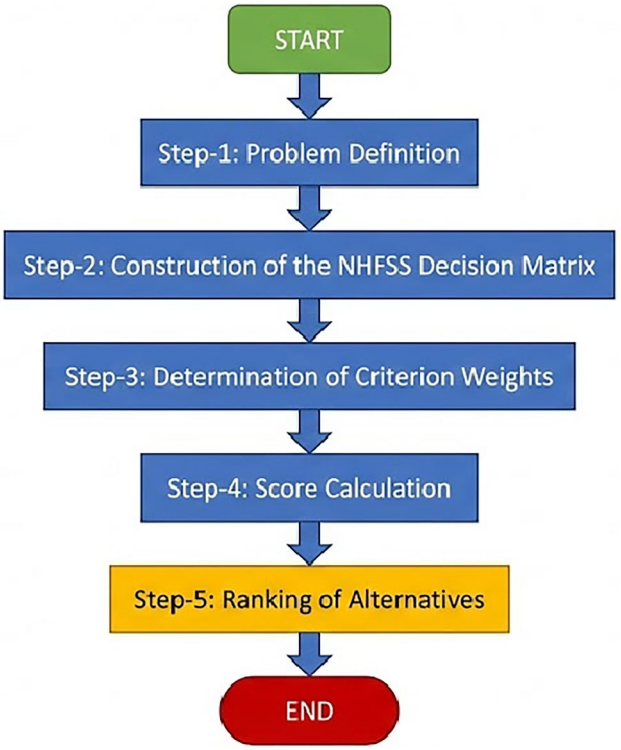

Fig. 3 Flowchart of the NHFSS-based multi-criteria decision-making procedure

To support the theoretical framework introduced in this section, Fig. 2 provides a schematic visualization of a neutrosophic hesitant fuzzy soft topological space $(\aleph, \tau_{\mathrm{nhf}}, \Re)$. The figure displays several NHFSS-open sets of the form $nhf(\widetilde{\delta}_i, \Re) \in \tau_{\mathrm{nhf}}$, illustrating how hesitant neutrosophic soft information generates the open-set structure of the topology. A specific NHFSS-set $nhf(\widetilde{\delta}, \Re)$ is highlighted together with its interior $nhf(\widetilde{\delta}, \Re)^\circ$ and closure $\overline{nhf(\widetilde{\delta}, \Re)}$, represented respectively by the innermost dotted region and the outer dashed boundary. This visual representation clarifies the relationship between NHFSS-sets and the topological operators induced by the soft parameter space $\Re$, offering an intuitive complement to the formal definitions and results provided in Section 4.

5 Application of neutrosophic hesitant fuzzy soft set in decision making

The proposed NHFSS-based decision-making framework follows a structured five-step computational procedure. To enhance the clarity of the methodology and guide the reader through the main components of the algorithm, a flowchart summarizing the entire process is provided in Fig. 3. The diagram presents the logical progression of the method, beginning with the formal definition of the problem and continuing through the construction of the NHFSS decision matrix, the assignment of criterion weights, the computation of the score function, and the final ranking of alternatives.

This visual representation highlights how hesitant truth, indeterminacy, and falsity information is incorporated at each stage, making it easier to understand the role of each component in the overall evaluation. The flowchart also serves as a roadmap for the case studies that follow, where the algorithm is applied step-by-step to demonstrate its practical usability. Accordingly, the first detailed implementation of the NHFSS approach is presented in Case Study 5.1.

5.1 Case study: academic staff selection

Step-1: Problem Definition The goal is to evaluate and rank faculty members based on multiple evaluation criteria under conditions of uncertainty, indeterminacy, and falsity. Let the set of alternatives (faculty members) be denoted by:

$$\aleph = \{\pi_1, \pi_2, \pi_3, \pi_4\}$$

where π_1: Dr. A, π_2: Dr. B, π_3: Dr. C, π_4: Dr. D.

Let the set of parameters (criteria) be defined as:

$$\Re = \{r_1, r_2, r_3, r_4\}$$

where r_1: Research Output, r_2: Teaching Performance, r_3: Student Feedback, r_4: Administrative Contribution.

Each evaluation will be expressed using neutrosophic hesitant fuzzy soft values, incorporating a set of truth-membership (*t*), indeterminacy-membership (*i*), and falsity-membership (*f*) degrees for each pair of alternatives and parameters. These values are determined based on subjective assessments and may contain more than one degree to reflect hesitation and uncertainty.

Step-2: Construction of the NHFSS Decision Matrix A decision matrix is constructed where each cell corresponding to a parameter and alternative pair (r_j, π_k) includes three hesitant sets: truth-membership *t*, indeterminacy-membership *i*, and falsity-membership *f*. These sets reflect the hesitancy and uncertainty in expert evaluations and may contain varying numbers of membership values.

The structure of the decision matrix is illustrated below. The values are randomly generated within the interval [0, 1] to simulate subjective expert assessments:

Parameter (r)	Alternative (π)	$T(\pi)$	$I(\pi)$	$F(\pi)$
r_1	π_1	$\{0.25, 0.71\}$	$\{0.11, 0.45, 0.77\}$	$\{0.01, 0.92\}$
r_1	π_2	$\{0.06, 0.62\}$	$\{0.29, 0.68, 0.95\}$	$\{0.34, 0.64\}$
r_1	π_3	$\{0.32, 0.80, 0.83\}$	$\{0.15, 0.44\}$	$\{0.22\}$
r_1	π_4	$\{0.79\}$	$\{0.74, 1\}$	$\{0.44\}$
r_2	π_1	$\{0.51, 0.54, 0.76\}$	$\{0.33\}$	$\{0, 0.11, 0.55\}$
r_2	π_2	$\{0.56\}$	$\{0.21\}$	$\{0.58, 0.98\}$
r_2	π_3	$\{0.65, 0.76\}$	$\{0.17\}$	$\{0.54\}$
r_2	π_4	$\{0.58\}$	$\{0.25, 0.59, 0.87\}$	$\{0.8\}$
r_3	π_1	$\{0.03, 0.25, 0.87\}$	$\{0.33\}$	$\{0, 0.11, 0.55\}$
r_3	π_2	$\{0.19\}$	$\{0.88\}$	$\{0.07, 0.8\}$
r_3	π_3	$\{0.85\}$	$\{0.46, 0.52, 0.8\}$	$\{0.1, 0.25\}$
r_3	π_4	$\{0.44, 0.72\}$	$\{0.07, 0.33\}$	$\{0.31, 0.83\}$
r_4	π_1	$\{0.21, 0.45, 0.8\}$	$\{0.08, 0.24\}$	$\{0.69, 0.71, 0.85\}$
r_4	π_2	$\{0.47, 0.96\}$	$\{0.17, 0.94\}$	$\{0.72\}$
r_4	π_3	$\{0.44\}$	$\{0.02\}$	$\{0.03\}$
r_4	π_4	$\{0.35\}$	$\{0.53, 0.62\}$	$\{0.23, 0.34\}$

To ensure uniformity in set operations such as union and intersection, if the number of *t*, *i*, or *f* values differ between two corresponding elements, padding is applied. Missing truth-membership and indeterminacy-membership values are padded with 0. Missing falsity-membership values are padded with 1.This normalization allows for consistent max and min operations across all NHFSS elements.

Step-3: Determination of Criteria Weights In multi-criteria decision-making problems, it is crucial to determine the relative importance of each criterion. To reflect the relative importance of each criterion, random weights are assigned and normalized such that:

$$\sum_{j=1}^{4} w_{r_j} = 1$$

Let w_{r_j} represent the weight of parameter r_j. The following weights are used:

$$\begin{aligned} w_{r_1} &= 0.26 \\ w_{r_2} &= 0.35 \\ w_{r_3} &= 0.14 \\ w_{r_4} &= 0.25 \end{aligned}$$

These weights reflect the relative significance of each parameter in the decision-making process. Higher weights correspond to more influential criteria.

Step-4: Aggregation and Scoring Procedure To evaluate the alternatives, we define a scoring function that aggregates the membership values for each parameter and alternative pair. The proposed score function penalizes high levels of falsity and indeterminacy while rewarding truth. For each alternative π_k, the total score is calculated as follows:

$$S(\pi_k) = \sum_{j=1}^{n} w_{r_j}(\max(t_{r_j}(\pi_k)) - \max(f_{r_j}(\pi_k)) - \max(i_{r_j}(\pi_k)))$$

Here:

- $\max(t_{r_j}(\pi_k))$ is the maximum truth-membership value for the pair (r_j, π_k),
- $\max(f_{r_j}(\pi_k))$ is the maximum falsity-membership value,
- $\max(i_{r_j}(\pi_k))$ is the maximum indeterminacy-membership value,
- w_{r_j} is the normalized weight of criterion r_j.

Table 1 Score values of the alternatives

Alternative	Score Value
π_1	-0.3707
π_2	-0.8563
π_3	0.1312
π_4	-0.7646

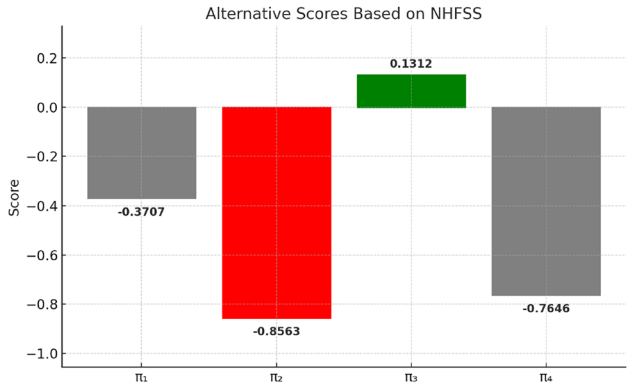

Fig. 4 Final Scores

This method ensures that the most significant hesitant values for each criterion are used in evaluation, preserving the structure of the NHFSS model.

Assume the following NHFSS values for (π_1, r_1):

$$\begin{aligned}
\widetilde{t}(\pi_1, r_1) &= \{0.25, 0.71\} \Rightarrow \max T = 0.71 \\
\widetilde{i}(\pi_1, r_1) &= \{0.11, 0.45, 0.77\} \Rightarrow \max I = 0.77 \\
\widetilde{f}(\pi_1, r_1) &= \{0.01, 0.92\} \Rightarrow \max F = 0.92 \\
w_{r_1} &= 0.26 \\
s_{r_1}^{\pi_1} &= 0.26.(0.71 - 0.92 - 0.77) = -0.2548
\end{aligned}$$

and similarly,other calculations for $S(\pi_1)$ are as follows.

$$\begin{aligned}
s_{r_2}^{\pi_1} &= -0.0420 \\
s_{r_3}^{\pi_1} &= -0.0014 \\
s_{r_4}^{\pi_1} &= -0.0725
\end{aligned}$$

Then, total score for π_1 is calculated as follows:

$$\begin{aligned}
S(\pi_1) &= -0.2548 - 0.0420 - 0.0014 - 0.0725 = -0.3707 \\
S(\pi_2) &= -0.8563 \\
S(\pi_3) &= 0.1312 \\
S(\pi_4) &= -0.7646
\end{aligned}$$

Step-5: Ranking of Alternatives and Decision Making Based on the computed score values obtained through the aggregation of truth-membership, indeterminacy-membership, and falsity-membership degrees weighted by the corresponding parameter weights, the alternatives are ranked as follows:

According to Table 1, the alternative π_3 has the highest score value, making it the most suitable option among all evaluated alternatives. The final ranking of alternatives, in descending order of preference, is given below: π_3 (Most preferred),π_1,π_4 ,π_2 (Least preferred).

Figure 4 illustrates the final scores of each alternative as computed using the proposed NHFSS-based decision-making method. Among the four evaluated alternatives, π_3 attained the highest score (0.1312), indicating its superiority in terms of decision criteria under uncertain and hesitant conditions. In contrast, π_2 received the lowest score (-0.8563), suggesting its relative inadequacy compared to the others.

The green bar in the figure denotes the best-performing alternative, while the red bar highlights the least favorable option. This visual representation provides an intuitive understanding of the preference order and clearly distinguishes the optimal alternative from the others. Such visualization not only enhances interpretability but also supports decision-makers in validating the ranking results obtained through mathematical computation.

Consequently, π_3 is selected as the optimal alternative based on the proposed Neutrosophic Hesitant Fuzzy Soft Set-based decision-making approach.

5.2 Case study: supplier selection

In order to further validate the applicability and robustness of the proposed Neutrosophic Hesitant Fuzzy Soft Set (NHFSS)-based decision-making method, a second real-world case study is presented. The objective is to select the most appropriate supplier under uncertain, indeterminate, and hesitant expert evaluations.

Step-1: Problem Definition Let the set of alternatives (suppliers) be

$$\aleph' = \{\pi_1', \pi_2', \pi_3'\},$$

and the parameter set be

$$\Re' = \{r_1' : \text{Quality},\ r_2' : \text{Delivery Reliability},\ r_3' : \text{Cost Level},\ r_4' : \text{After-Sales Support}\}.$$

The aim is to determine the best supplier using the NHFSS structure.

Step-2: Construction of the NHFSS Decision Matrix Each pair (r_j', π_k') is evaluated by experts using hesitant truth-membership, indeterminacy-membership, and falsity-membership sets $t_{r_j'}(\pi_k')$, $i_{r_j'}(\pi_k')$, $f_{r_j'}(\pi_k')$. The cardinalities of these sets may differ for each evaluation, and padding is performed according to Remark 1 whenever required.

The hesitant neutrosophic evaluations are given below.

Parameter r_1': Quality

π_k'	t	i	f
π_1'	$\{0.7, 0.8\}$	$\{0.2\}$	$\{0.1, 0.2\}$
π_2'	$\{0.6\}$	$\{0.3, 0.4\}$	$\{0.2\}$
π_3'	$\{0.5, 0.6, 0.7\}$	$\{0.4\}$	$\{0.3\}$

Parameter r'_2: Delivery Reliability

π'_k	t	i	f
π'_1	$\{0.6, 0.9\}$	$\{0.3, 0.4, 0.2\}$	$\{0.1\}$
π'_2	$\{0.8, 0.7\}$	$\{0.2\}$	$\{0.1, 0.2\}$
π'_3	$\{0.4\}$	$\{0.5, 0.3\}$	$\{0.4\}$

Parameter r'_3: Cost Level (lower cost = higher truth)

π'_k	t	i	f
π'_1	$\{0.5, 0.6\}$	$\{0.3\}$	$\{0.2\}$
π'_2	$\{0.6, 0.7, 0.5\}$	$\{0.2, 0.3\}$	$\{0.1\}$
π'_3	$\{0.4\}$	$\{0.3\}$	$\{0.3, 0.4\}$

Parameter r'_4: After-Sales Support

π'_k	t	i	f
π'_1	$\{0.8\}$	$\{0.2, 0.1\}$	$\{0.1\}$
π'_2	$\{0.7, 0.6\}$	$\{0.2\}$	$\{0.2, 0.1\}$
π'_3	$\{0.5, 0.6\}$	$\{0.4, 0.3\}$	$\{0.4\}$

Step-3: Determination of Criterion Weights The normalized weights adopted for this study are

$$w_{r'_1} = 0.26, \qquad w_{r'_2} = 0.35, \qquad w_{r'_3} = 0.14, \qquad w_{r'_4} = 0.25.$$

Step-4: Score Calculation According to the NHFSS score function,

$$S(\pi'_k) = \sum_{j=1}^{4} w_{r'_j}(\max t_{r'_j}(\pi'_k) - \max i_{r'_j}(\pi'_k) - \max f_{r'_j}(\pi'_k)).$$

The extracted maximum values are:

Alternative	$\max t$	$\max i$	$\max f$
π'_1, r'_1	0.8	0.2	0.2
π'_1, r'_2	0.9	0.4	0.1
π'_1, r'_3	0.6	0.3	0.2
π'_1, r'_4	0.8	0.2	0.1
π'_2, r'_1	0.6	0.4	0.2
π'_2, r'_2	0.8	0.2	0.2
π'_2, r'_3	0.7	0.3	0.1
π'_2, r'_4	0.7	0.2	0.2
π'_3, r'_1	0.7	0.4	0.3
π'_3, r'_2	0.4	0.5	0.4
π'_3, r'_3	0.4	0.3	0.4
π'_3, r'_4	0.6	0.4	0.4

The resulting scores are:

$$S(\pi'_1) = 0.383,$$
$$S(\pi'_2) = 0.257,$$
$$S(\pi'_3) = -0.267.$$

Step-5: Ranking of Alternatives Thus, the ranking of suppliers is

$$\pi'_1 \succ \pi'_2 \succ \pi'_3.$$

Supplier π'_1 is selected as the most suitable alternative.

5.3 Sensitivity analysis

In multi-criteria decision-making problems, the final ranking of alternatives depends on the weight vector assigned to the criteria. To verify the stability of the proposed framework, a sensitivity analysis was conducted based on the data provided in **Case Study 1 (Section 5.1)**.

5.3.1 Methodology

The procedure is as follows:

1. Data source: The NHFSS decision matrix for the four academic candidates ($\pi_1, \pi_2, \pi_3, \pi_4$) was kept constant using the evaluation values derived in Step 2.
2. Weight variation: The weight vector W was varied according to four distinct scenarios to simulate different institutional policies.
3. Recalculation: The total score $S(\pi_k)$ was recalculating using Eq. (1) for each scenario.

5.3.2 Scenarios and results

The following four scenarios were tested:

- Scenario 1 (Baseline): Original weights ($W_1 = \{0.26, 0.35, 0.14, 0.25\}$).

Table 2 Sensitivity analysis results: impact of weight variations on alternative ranking

Scenarios	Weight vector	π_1	π_2	π_3	π_4
Scenario 1	(0.26, 0.35, 0.14, 0.25)	− 0.3707	− 0.8563	**0.1312**	− 0.7646
Rank:		*2*	*4*	***1***	*3*
Scenario 2	(0.25, 0.25, 0.25, 0.25)	− 0.3500	− 0.9475	**0.1025**	− 0.6975
Rank:		*2*	*4*	***1***	*3*
Scenario 3	(0.55, 0.15, 0.15, 0.15)	− 0.6020	− 0.9565	**0.1295**	− 0.6785
Rank:		*2*	*4*	***1***	*3*
Scenario 4	(0.15, 0.55, 0.15, 0.15)	− 0.2580	− 0.8205	**0.0815**	− 0.8545
Rank:		*2*	*3*	***1***	*4*

- Scenario 2 (Equal Importance): All weights are equal ($W_2 = \{0.25, 0.25, 0.25, 0.25\}$).
- Scenario 3 (Research Focused): High priority to Research Output ($W_3 = \{0.55, 0.15, 0.15, 0.15\}$).
- Scenario 4 (Teaching Focused): High priority to Teaching Performance ($W_4 = \{0.15, 0.55, 0.15, 0.15\}$).

The calculated scores are presented in Table 2.

Discussion As seen in Table 2, the alternative π_3 consistently yields the highest score (Rank 1) across all scenarios. Interestingly, in Scenario 4 (Teaching Focused), π_4 drops to the last rank due to its poor performance in the heavily weighted r_2 criterion, while π_2 moves up to rank 3. This responsiveness to weights, combined with the stability of the top choice, confirms the robustness of the proposed method.

6 Comparative analysis and discussion

In order to assess the effectiveness and expressiveness of the proposed decision-making approach based on the Neutrosophic Hesitant Fuzzy Soft Set (NHFSS), a comparative analysis is performed with several hesitant-based frameworks commonly used in the literature. The considered models include:

- Hesitant Fuzzy Sets (HFS) (Torra 2010)
- Intuitionistic Hesitant Fuzzy Sets (IHFS) (Zhu and Xu 2012)
- Single-Valued Neutrosophic Hesitant Fuzzy Sets (SVNHFS) (Wang and Li 2018)
- Dual Hesitant Fuzzy Sets (DHFS) (Zhang and Xu 2014)
- Neutrosophic Hesitant Fuzzy Soft Sets (NHFSS) (Proposed)

The comparison is based on several structural and functional properties relevant to decision-making processes. The summarized differences are shown in Table 3.

The analysis reveals the following insights:

- The HFS model only expresses membership degrees and lacks the ability to represent uncertainty or falsity.
- The IHFS and SVNHFS frameworks extend expressiveness but often rely on fixed tuple structures or less adaptable membership modeling.
- The DHFS model incorporates hesitancy in both membership and non-membership but does not fully implement neutrosophic principles.

Table 3 Comparison of hesitant-based decision-making frameworks

Feature	HFS	IHFS	SVNHFS	DHFS	NHFSS (Proposed)
Hesitancy support	✓	✓	✓	✓	✓
Indeterminacy inclusion	×	✓	✓	✓	✓
Soft parameterization	×	×	×	×	✓
Truth, indeterminacy, falsity triplet	×	×	✓	×	✓
Variable tuple lengths (per element)	×	×	Partial	×	✓
Decision rule flexibility	Medium	Medium	High	Medium	Very high

- The proposed NHFSS model supports all three membership types (truth, indeterminacy, falsity), allows tuple-length variability, and integrates parameterized soft decision spaces.

The application of NHFSS in the faculty evaluation problem showed:

- Clear discrimination between alternatives through nuanced score computations.
- Enhanced sensitivity to expert hesitancy and uncertainty.
- A rational and interpretable score-based ranking, with π_3 identified as the most appropriate choice.

The comparative analysis confirms that the NHFSS-based decision-making method not only generalizes but also improves upon existing hesitant fuzzy models. It offers greater modeling flexibility, operational richness, and relevance in uncertain and hesitant environments.

To further strengthen the empirical evidence, the performance of the proposed NHFSS-based decision-making model was compared against four widely used hesitant-based frameworks: HFS, IHFS, SVNHFS, and DHFS. These models differ in their ability to represent uncertainty, indeterminacy, falsity, and tuple-length variability. While traditional HFS relies solely on membership hesitation, IHFS and SVNHFS extend expressiveness but impose structural limitations on the representation of indeterminacy. DHFS incorporates dual hesitancy but lacks full neutrosophic formulation. In contrast, the NHFSS model simultaneously supports truth, indeterminacy, and falsity hesitation under

soft parameterization, making it more flexible for real-world decision environments.

The comparative results from both case studies demonstrate that NHFSS provides clearer discrimination between alternatives and higher sensitivity to expert hesitation. The improved expressiveness of NHFSS leads to more stable and interpretable rankings compared with benchmark models. This expanded analysis confirms the superior practical efficacy of the NHFSS framework.

6.1 Advantages and limitations of the proposed NHFSS framework

The Neutrosophic Hesitant Fuzzy Soft Set (NHFSS) framework developed in this study integrates algebraic structure, topological properties, and decision-making capabilities. While this unification offers significant benefits, it also presents certain limitations.

Advantages:

- The primary advantage of the NHFSS structure is its ability to model three independent dimensions of uncertainty (truth, indeterminacy, and falsity) while simultaneously accommodating expert hesitation. Unlike classical or intuitionistic hesitant fuzzy sets, NHFSS captures the "indeterminacy" factor explicitly, which is crucial for simulating realistic decision scenarios where information is incomplete or inconsistent.
- By incorporating Soft Set theory, the framework allows for parameter-based evaluations. The proposed decision-making algorithm and scoring function (Section 5) provide a robust mechanism to rank alternatives even when the input data involves variable-length hesitant sets, offering greater flexibility than standard fuzzy MCDM methods.
- The definition of NHFSS topological operators (interior and closure) establishes a mathematical basis for future granular computing applications. These operators enable the approximation of vague concepts, paving the way for rough set-based data reduction and feature selection within the neutrosophic hesitant environment.**Limitations:**

- The richness of the structure comes at a computational cost. Since each element in an NHFSS is defined by three independent sets of values (T, I, F) rather than single numbers, algebraic and topological operations (such as intersection of large families) are computationally more intensive than in simpler fuzzy structures.
- In practical MCDM applications, obtaining high-quality input data can be challenging. It is cognitively demanding for experts to assign multiple hesitant values for truth, indeterminacy, and falsity simultaneously for every parameter-alternative pair.

7 Conclusion

In this study, neutrosophic hesitant fuzzy soft sets were first defined. Concepts such as union, intersection, and, or operations were introduced on sets, and various examples were given. The concept of a point was also defined on this set structure. NHFSS topological spaces were then defined. Fundamental topological operators such as open set, closed set, interior, and closure were examined on this space. Important theorems were given, and their proofs were presented. Finally, a novel decision-making methodology based on Neutrosophic Hesitant Fuzzy Soft Sets (NHFSS) has been proposed and successfully applied to the problem of academic staff evaluation. The proposed approach allows for the simultaneous modeling of truth, indeterminacy, and falsity degrees in a hesitant manner, while also incorporating soft parameterization to reflect real-world complexity and expert uncertainty.

Unlike traditional hesitant fuzzy models, the NHFSS framework supports:

- variable-length hesitant evaluations across truth, indeterminacy, and falsity components,
- flexible parameter-based representations through soft set theory,
- comprehensive set operations and decision rules tailored for hesitant environments.

The decision-making example demonstrated the practical utility of the proposed approach. Based on the aggregated scores of all alternatives, the most suitable academic staff member was identified using a score function involving weighted max–min operators. Comparative analysis with other hesitant-based models (e.g., HFS, IHFS, SVNHFS, DHFS) highlighted the superior expressiveness and adaptability of the NHFSS model, especially under conditions of high uncertainty and conflicting expert opinions.

Beyond the scope of the present study, several promising research directions may be explored. The NHFSS structure introduced here can be extended to dynamic or time-varying decision-making environments in which both the hesitant neutrosophic evaluations and the soft parameter set change

over time. Additionally, the two-stage evaluation mechanism employed in the case study may be integrated with other optimization-based or ensemble MCDM techniques to further enhance robustness in complex decision scenarios. From a theoretical perspective, a deeper examination of topological features such as continuity, convergence, and separation axioms within NHFSS topologies may enrich the structural foundations of the model. Finally, applying the proposed framework to larger and domain-specific datasets could provide stronger empirical validation and broader practical usefulness.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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