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Unpredictability of initial coefficient bounds for *m*-fold symmetric bi-univalent starlike and convex functions defined by subordinations

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Abstract In this paper, we introduce certain new subclasses of the bi-univalent function class σ in which both f and f^{-1} are m-fold symmetric analytic with their derivatives in the class $\mathcal{P}$ of analytic functions. Furthermore, we obtain coefficient bounds of $|a_{m+1}|$ and $|a_{2m+1}|$ for these new subclasses.

Keywords Analytic functions · Univalent functions · Bi-univalent functions · Taylor–Maclaurin coefficients · m-fold symmetric functions · Subordination

Mathematics Subject Classification Primary 30C45 · 33C50; Secondary 30C80

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}, \tag{1.1}$$

which are *analytic* in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. For more details on univalent functions, see [8]. For $0 \leq \beta < 1$, let $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ be the subclasses of $\mathcal{S}$ consisting from *starlike* functions

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of order β and *convex* functions of order β, respectively. Their analytic descriptions are

$$\mathcal{S}^*(\beta)=\left\{f: f\in\mathcal{S},\ \Re\left(\frac{zf'(z)}{f(z)}\right)>\beta\quad(z\in\mathbb{U})\right\}, \tag{1.2}$$

and

$$\mathcal{C}(\beta)=\left\{f: f\in\mathcal{S},\ \Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta\quad(z\in\mathbb{U})\right\}, \tag{1.3}$$

respectively. These functions are typically characterized by the quantity $zf'(z)/f(z)$ or 1 + $zf''(z)/f'(z)$ lying in a certain domain starlike with respect to 1 in the right-half plane. Subordination is useful to unify these subclasses. An analytic function f is subordinate to an analytic function g, written $f(z)\prec g(z)$, if there exists an analytic function w with $w(0)=0$ and $|w(z)|<1$ for $z\in\mathbb{U}$ such that $f(z)=g(w(z))$. Ma and Minda [13] unified various subclasses of starlike and convex functions for which either of the quantity $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\mathbb{U}$, $\varphi(0)=1$, $\varphi'(0)>0$, and φ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes of Ma–Minda starlike and convex functions consists of functions $f\in\mathcal{A}$ satisfying the subordination $zf'(z)/f(z)\prec\varphi(z)$ and $1+zf''(z)/f'(z)\prec\varphi(z)$. It is well known that every function $f\in\mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z,\quad(z\in\mathbb{U}) \tag{1.4}$$

and

$$f(f^{-1}(w))=w,\left(|w|<r_0(f);r_0(f)\geq\frac{1}{4}\right). \tag{1.5}$$

Indeed, the inverse function may have an analytic continuation to $\mathbb{U}$, with

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.6}$$

Let $k(z)=k_1z+k_2z^2+\cdots$ and $\ell(z)=\ell_1z+\ell_2z^2+\cdots$, with $k_1=\ell_1$ two functions mapping the open unit disk onto a univalent domain, it would imply that $z\prec k(z)$ and so that k is trivial. Then the function $f(z)=k\left[\ell^{-1}(z)\right]$ is bi-univalent. Let σ denote the class of *bi-univalent* functions in $\mathbb{U}$, given by equation (1.1). A function f is bi-starlike of Ma–Minda type or bi-convex of Ma–Minda type if both f and f^{-1} are respectively Ma–Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\sigma(\varphi)$ and $\mathcal{C}_\sigma(\varphi)$. Lewin [12] investigated the class of *bi-univalent* functions σ and obtained a bound $|a_2|\leq 1.51$. Motivated by the work of Lewin [12], Brannan and Taha found non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ of the functions in the class $\mathcal{S}^*_\sigma(\beta)$ and also in $\mathcal{C}_\sigma(\beta)$ (for details see [3]). Brannan and Clunie [2] conjectured that $|a_2|\leq\sqrt{2}$. The coefficient estimate problem for each of the following Taylor–Maclaurin coefficients: $|a_n|\quad(n\in\mathbb{N},\quad n\geqq 3)$ is still open. In recent times, the study of *bi-univalent* functions gained momentum mainly due to the work of Srivastava et al. [20]. Motivated by this, many researchers (see [4–7,9–11,14–16,18,20–22]) recently investigated several interesting subclasses of the class σ and found non-sharp estimates on the first two Taylor–Maclaurin coefficients. Further, in a recent investigation, Srivastava et. al. [19] has provided few examples for the class of m-fold symmetric bi-univalent functions and as an application found the coefficient estimates for $|a_{m+1}|$ and $|a_{2m+1}|$ for a new subclass of functions. For each function $f\in\mathcal{S}$, the function $h(z)=\sqrt[m]{f(z^m)}$ ($z\in\mathbb{U}$, $m\in\mathbb{N}$) is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [17]) if it has the normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in \mathbb{U}. \tag{1.7}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$ which are normalized by the above series expansion. In fact, the functions in the class $\mathcal{S}$ are one fold symmetric. Analogous to the concept of m-fold symmetric univalent functions, we here introduce the concept of m-fold symmetric *bi-univalent* functions. Each function $f \in \sigma$, generates an m-fold symmetric *bi-univalent* function for each integer m. The normalized form of f is given as in (1.7) and f^{-1} is given as follows

$$\begin{aligned} g(w) = {} & w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \cdots \end{aligned} \tag{1.8}$$

where $f^{-1} = g$. We denote by σ_m the class of m-fold symmetric *bi-univalent* functions in $\mathbb{U}$. For $m = 1$, the formula (1.8) coincides with the formula (1.6) of the class σ. Also, we denote by $\mathcal{P}$, the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ such that $\operatorname{Re}(p(z)) > 0$ in $\mathbb{U}$. In view of Pommerenke [17], the m-fold symmeric functions in the class $\mathcal{P}$ is of the form

$$p(z) = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots$$

The objective of the present paper is to introduce several new subclasses of *bi-univalent* Ma–Minda starlike and convex functions in which both f and f^{-1} are m-fold symmetric analytic functions with derivative in $\mathcal{P}$ and obtain coefficient bounds of $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses.

2 Coefficient bound for the function class $H_{\sigma,m}(\varphi)$

Let φ be an analytic function with positive real part in $\mathbb{U}$ such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_i \in \mathbb{C} \text{ for } i \in \{2, 3, \ldots\}, \ B_1 > 0). \tag{2.1}$$

Definition 2.1 A function $f(z)$, given by (1.7), is said to be in the class $H_{\sigma,m}(\varphi)$ if the following conditions are satisfied :

$$f \in \sigma_m, \ f'(z) \prec \varphi(z) \ \textit{and} \ g'(w) \prec \varphi(w), \ g(w) = f^{-1}(w)$$

where the function g is defined by (1.8).

For the special choices of the function $\varphi(z)$ and for the choice of $m = 1$, our subclass reduces to the following.

(1) For $m = 1$, $H_{\sigma,m}(\varphi) \equiv H_{\sigma,1}(\varphi) = H_\sigma(\varphi)$ studied by Ali et al. [1].

(2) For $m = 1$, and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\gamma$ $(0 \le \gamma < 1)$, $H_{\sigma,m}(\varphi) \equiv H_{\sigma,1}\left(\left(\frac{1+z}{1-z}\right)^\gamma\right)$ studied by Srivastava et al. [20].

(3) For $m = 1$, and $\varphi(z) = \left(\frac{1+(1-2\alpha)z}{1-z}\right)(0 \leq \alpha < 1)$, $H_{\sigma,m}(\varphi) \equiv H_{\sigma,1}\left(\frac{1+(1-2\alpha)z}{1-z}\right)$ studied by Srivastava et al. [20].

Theorem 2.1 *Let $f(z)$, given by (1.7), be the class $H_{\sigma,m}(\varphi)$. Then*

$$|a_{m+1}| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{(m+1)\left|\left[(2m+1)B_1^2 + 2(1+m)(B_1 - B_2)\right]\right|}}, \tag{2.2}$$

and

$$|a_{2m+1}| \leq \left(\frac{1}{(1+2m)} + \frac{B_1}{2(1+m)}\right)B_1. \tag{2.3}$$

Proof Let $f \in H_{\sigma,m}(\varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$, with $u(0) = v(0) = 0$ satisfying

$$f'(z) = \varphi(u(z)) \text{ and } g'(w) = \varphi(v(w)). \tag{2.4}$$

By definition of the functions p_1 and p_2

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots,$$
$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \cdots$$

or equivalently

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_m z^m + \left(c_{2m} - \frac{c_m^2}{2}\right)z^{2m} + \cdots\right) \tag{2.5}$$

and

$$v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left(b_m z^m + \left(b_{2m} - \frac{b_m^2}{2}\right)z^{2m} + \cdots\right). \tag{2.6}$$

Then p_1 and p_2 are analytic in $\mathbb{U}$ with $p_1(0) = p_2(0) = 1$. Since $u, v : \mathbb{U} \to \mathbb{U}$, the functions p_1 and p_2 have positive real part in $\mathbb{U}$ and $|c_k| \leq 2$ and $|b_k| \leq 2$, for all k.

From (2.4), (2.5) and (2.6),

$$f'(z) = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{2.7}$$

and

$$g'(w) = \varphi\left(\frac{p_2(w) - 1}{p_2(w) + 1}\right). \tag{2.8}$$

Using (2.5) and (2.6), together with (2.1) we get

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1 c_m z^m + \left(\frac{1}{2}B_1\left(c_{2m} - \frac{c_m^2}{2}\right) + \frac{1}{4}B_2 c_m^2\right)z^{2m} + \cdots \tag{2.9}$$

and

$$\varphi\left(\frac{p_2(w) - 1}{p_2(w) + 1}\right) = 1 + \frac{1}{2}B_1 b_m w^m + \left(\frac{1}{2}B_1\left(b_{2m} - \frac{b_m^2}{2}\right) + \frac{1}{4}B_2 b_m^2\right)w^{2m} + \cdots \tag{2.10}$$

Since

$$f'(z) = 1 + (m+1)\,a_{m+1}z^{m} + (2m+1)\,a_{2m+1}z^{2m} + \cdots,$$

and

$$g'(w) = 1 - (m+1)\,a_{m+1}w^{m} + (2m+1)\left[(m+1)\,a_{m+1}^{2} - a_{2m+1}\right]w^{2m} + \cdots$$

it follows from (2.7), (2.8), (2.9) and (2.10) that

$$(m+1)\,a_{m+1} = \frac{1}{2}B_1c_m, \tag{2.11}$$

$$(2m+1)\,a_{2m+1} = \frac{1}{2}B_1\left(c_{2m} - \frac{c_m^2}{2}\right) + \frac{1}{4}B_2c_m^2, \tag{2.12}$$

$$-(m+1)\,a_{m+1} = \frac{1}{2}B_1b_m, \tag{2.13}$$

and

$$(2m+1)\left[(m+1)\,a_{m+1}^{2} - a_{2m+1}\right] = \frac{1}{2}B_1\left(b_{2m} - \frac{b_m^2}{2}\right) + \frac{1}{4}B_2b_m^2. \tag{2.14}$$

From (2.11) and (2.13), we have

$$c_m = -b_m. \tag{2.15}$$

Also from (2.12), (2.13), (2.14) and (2.15), we obtain

$$a_{m+1}^{2} = \frac{B_1^3\,(c_{2m} + b_{2m})}{2\,(m+1)\left((2m+1)\,B_1^2 + 2\,(m+1)\,(B_1 - B_2)\right)}.$$

Thus, in view of the inequalities $|c_{2m}| \le 2$ and $|b_{2m}| \le 2$ for $|a_{m+1}|$,we get the desired result as in (2.2). By subtracting (2.14) from (2.12) and further computations using (2.11) and (2.15) gives us

$$a_{2m+1} = \frac{B_1\,(c_{2m} - b_{2m})}{4\,(2m+1)} + \frac{B_1^2c_m^2}{8\,(1+m)},$$

which, in view of the inequalities $|c_{2m}| \le 2$, $|c_m| \le 2$ and $|b_{2m}| \le 2$ for $|a_{2m+1}|$, we get the desired result as in (2.3). This completes the proof of Theorem 2.1. □

3 Coefficient bound for the function class $\mathcal{M}_{\sigma,m}(\lambda,\varphi)$

Definition 3.1 A function $f(z)$, given by (1.7), is said to be in the class $\mathcal{M}_{\sigma,m}(\lambda,\varphi)$ if the following conditions are satisfied:

$$f \in \sigma_m,\ \ (1-\lambda)\,\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)$$

and

$$(1-\lambda)\,\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w),$$

where the function g is defined by (1.8).

For $m = 1$ and the special choices of the function $\varphi(z)$ and λ, our subclass reduces to the following.

(1) For $m = 1$, $\mathcal{M}_{\sigma,m}(\lambda,\varphi) \equiv \mathcal{M}_{\sigma,1}(\lambda,\varphi) = \mathcal{M}_{\sigma}(\varphi)$ studied by Ali et al. [1].
(2) For $m = 1$, $\lambda = 1$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$ $(0 < \gamma \leq 1)$, $\mathcal{M}_{\sigma,1}\left(1, \left(\frac{1+z}{1-z}\right)^{\gamma}\right) \equiv \mathcal{ST}_{\sigma}^{*}(\gamma)$, the class of strongly bi-starlike functions of order γ studied by Brannan and Taha [3].
(3) For $m = 1$, $\lambda = 0$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$ $(0 < \gamma \leq 1)$, $\mathcal{M}_{\sigma,1}\left(1, \left(\frac{1+z}{1-z}\right)^{\gamma}\right) \equiv \mathcal{CV}_{\sigma}(\gamma)(0 < \gamma \leq 1)$, the class of strongly bi-convex functions of order γ studied by Brannan and Taha [3].
(4) For $m = 1$, $\lambda = 1$ and $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}(0 \leq \alpha < 1)$, $\mathcal{M}_{\sigma,1},\left(1, \left(\frac{1+(1-2\alpha)z}{1-z}\right)\right) \equiv \mathcal{C}_{\sigma}(\alpha)(0 \leq \alpha < 1)$, the class of bi-convex functions of order α studied by Brannan and Taha [3].
(5) For $m = 1$, $\lambda = 0$ and $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}(0 \leq \alpha < 1)$, $\mathcal{M}_{\sigma,1}\left(1, \left(\frac{1+(1-2\alpha)z}{1-z}\right)\right) \equiv \mathcal{S}_{\sigma}^{*}(\alpha)(0 \leq \alpha < 1)$, the class of bi-starlike functions of order α studied by Brannan and Taha [3].

For functions in the class $M_{\sigma,m}(\lambda,\varphi)$, the following coefficient estimates hold.

Theorem 3.1 *Let $f(z)$, given by (1.7), be the class $M_{\sigma,m}(\lambda,\varphi)$,$\lambda \geq 0$ Then*

$$|a_{m+1}| \leq \frac{B_1\sqrt{B_1}}{m\sqrt{\left|(1+\lambda m)\left(B_1^2 + (1+\lambda m)(B_1 - B_2)\right)\right|}}, \tag{3.1}$$

and

$$|a_{2m+1}| \leq \frac{\left(1 + m + 2\lambda m + 2\lambda m^2\right)(B_1 + |B_2 - B_1|)}{2m^2(1+\lambda m)(1+2\lambda m)}. \tag{3.2}$$

Proof Let $f \in M_{\sigma,m}(\lambda,\varphi)$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$, with $u(0) = v(0) = 0$ satisfying

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(u(z)) \tag{3.3}$$

and

$$(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi(v(w)). \tag{3.4}$$

Since

$$\begin{aligned}(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) &= 1 + m(1+\lambda m)a_{m+1}z^m \\ &+ \left[2m(1+2\lambda m)a_{2m+1} - m\left(1 + 2\lambda m + \lambda m^2\right)a_{m+1}^2\right]z^{2m} + \cdots\end{aligned}$$

and

$$\begin{aligned}&(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) \\ &= 1 - m(1+\lambda m)a_{m+1}w^m + \left[m\left(1 + 2\lambda m + 2m + 3\lambda m^2\right)a_{m+1}^2\right. \\ &\quad \left.-2m(1+2\lambda m)a_{2m+1}\right]w^{2m} + \cdots\end{aligned}$$

then from (2.9), (2.10), (3.3) and (3.4) we get

$$m\left(1+\lambda m\right)a_{m+1}=\frac{1}{2}B_1c_m, \tag{3.5}$$

$$\begin{aligned}&2m\left(1+2\lambda m\right)a_{2m+1}-m\left(1+2\lambda m+\lambda m^2\right)a_{m+1}^2\\&\quad=\frac{1}{2}B_1\left(c_{2m}-\frac{c_m^2}{2}\right)+\frac{1}{4}B_2c_m^2,\end{aligned} \tag{3.6}$$

$$-m\left(1+\lambda m\right)a_{m+1}=\frac{1}{2}B_1b_m, \tag{3.7}$$

and

$$\begin{aligned}&m\left(1+2\lambda m+2m+3\lambda m^2\right)a_{m+1}^2-2m\left(1+2\lambda m\right)a_{2m+1}\\&\quad=\frac{1}{2}B_1\left(b_{2m}-\frac{b_m^2}{2}\right)+\frac{1}{4}B_2b_m^2,\end{aligned} \tag{3.8}$$

From (3.5) and (3.7), we obtain

$$c_m=-b_m. \tag{3.9}$$

and also from (3.6), (3.8) and (3.9), we have

$$a_{m+1}^2=\frac{B_1^3\left[c_{2m}+b_{2m}\right]}{4m^2\left(1+\lambda m\right)\left[B_1^2+\left(1+\lambda m\right)\left(B_1-B_2\right)\right]},$$

in view of the inequalities $|c_{2m}|\leq 2$ and $|b_{2m}|\leq 2$ for $|a_{m+1}|$, we get as asserted in(3.1). Next, to find the bound on $|a_{2m+1}|$, by a simple calculations using, (3.6), (3.8) and (3.9), we get

$$\begin{aligned}a_{2m+1}=&\frac{\frac{B_1}{2}\left\{\left(1+2\lambda m+2m+3\lambda m^2\right)c_{2m}+\left(1+2\lambda m+\lambda m^2\right)b_{2m}\right\}}{4m^2\left(1+2\lambda m\right)\left(1+\lambda m\right)}\\&+\frac{\frac{(B_2-B_1)}{2}\left(1+2\lambda m+m+2\lambda m^2\right)c_m^2}{4m^2\left(1+2\lambda m\right)\left(1+\lambda m\right)}\end{aligned}$$

which, in view of the inequalities $|c_m|\leq 2$ and $|b_{2m}|\leq 2$ for $|a_{2m+1}|$, we get our desired inequality as asserted in (3.2). □

For the case of one fold symmetric functions with $\lambda=0$ and $\lambda=1$, Theorem 3.1 reduces to the following Corollaries 3.2 and 3.3 stated below.

Corollary 3.2 *Let $f(z)$, given by (1.7), be in the class $\mathcal{S}_\sigma^*(\varphi)$. Then*

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2+\left(B_1-B_2\right)\right|}},$$

and

$$|a_3|\leq B_1+|B_2-B_1|.$$

Corollary 3.3 *Let $f(z)$, given by (1.7), be in the class and $\mathcal{C}_\sigma(\varphi)$. Then*

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\left|2B_1^2+4\left(B_1-B_2\right)\right|}},$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{2}.$$

4 Coefficient bound for the function class $\mathcal{N}_{\sigma,m}(\lambda,\varphi)$

Definition 4.1 A function $f(z)$, given by (1.7), is said to be in the class $\mathcal{N}_{\sigma,m}(\lambda,\varphi)$ if the following conditions are satisfied:

$$f \in \sigma_m, \frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} \prec \varphi(z), \lambda \geq 0$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)} \prec \varphi(w), \; g(w) = f^{-1}(w),$$

where the function g is defined by (1.8).

For $m = 1$, $\lambda = 0$ and the special choices of the function $\varphi(z)$, our subclass reduces to the following.

(1) For $m = 1$, $\mathcal{N}_{\sigma,m}(\lambda,\varphi) \equiv \mathcal{N}_{\sigma,1}(\lambda,\varphi)$ was studied by Ali et al. [1].

(2) For $m = 1$, $\lambda = 0$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$ $(0 \leq \gamma < 1)$, $\mathcal{N}_{\sigma,m}\left(0, \left(\frac{1+z}{1-z}\right)^{\gamma}\right) \equiv \mathcal{ST}_{\sigma}^{*}(\gamma)$ the class of bi-strongly starlike functions of order γ studied by Brannan and Taha [3].

(3) For $m = 1$, $\lambda = 0$ and $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$, $\mathcal{N}_{\sigma,m}\left(0, \frac{1+(1-2\alpha)z}{1-z}\right) \equiv \mathcal{S}_{\sigma}^{*}(\gamma)$ the class of bi-starlike functions of order γ studied by Brannan and Taha [3].

Theorem 4.1 *Let $f(z)$, given by (1.7), be the class $\mathcal{N}_{\sigma,m}(\lambda,\varphi)$, $\lambda \geq 0$. Then*

$$|a_{m+1}| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{\left|m(3\lambda m^2 + 4\lambda m + 2m^2 + \lambda)B_1^2 + 2m^2(1+\lambda+\lambda m)^2(B_1 - B_2)\right|}}, \tag{4.1}$$

and

$$|a_{2m+1}| \leq \frac{(m+1)(B_1 + |B_2 - B_1|)}{2m^2(1+2\lambda+2\lambda m)}. \tag{4.2}$$

Proof Let $f \in \mathcal{N}_{\sigma,m}(\lambda,\varphi)$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$, with $u(0) = v(0) = 0$ satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = \varphi(u(z)) \tag{4.3}$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)} = \varphi(v(w)), g(w) = f^{-1}(w). \tag{4.4}$$

Since

$$\begin{aligned}
&\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f(z)}{f(z)} \\
&\quad = 1 + m(1+\lambda(m+1))a_{m+1}z^m + (2m(1+\lambda(2m+1))a_{2m+1} \\
&\qquad - m(1+\lambda(m+1))a_{m+1}^2)z^{2m} + \cdots,
\end{aligned}$$

and

$$\frac{wg'(w)}{g(w)}+\frac{\lambda w^2 g''(w)}{g(w)}=1-m(1+\lambda(m+1))a_{m+1}w^m+\big((m(2m+1)+\lambda m(m+1)(4m+1))a_{m+1}^2-2m(1+\lambda(2m+1))a_{2m+1}\big)w^{2m}+\cdots.$$

then from (2.9), (2.10), (4.3) and (4.4), we have

$$m(1+\lambda(m+1))a_{m+1}=\frac{1}{2}B_1c_m, \tag{4.5}$$

$$2m(1+\lambda(2m+1))a_{2m+1}-m(1+\lambda(m+1))a_{m+1}^2=\frac{1}{2}B_1\left(c_{2m}-\frac{c_m^2}{2}\right)+\frac{1}{4}B_2c_m^2, \tag{4.6}$$

$$-m(1+\lambda(m+1))a_{m+1}=\frac{1}{2}B_1b_m, \tag{4.7}$$

and

$$(m(2m+1)+\lambda m(m+1)(4m+1))a_{m+1}^2-2m(1+\lambda(2m+1))a_{2m+1}=\frac{1}{2}B_1\left(b_{2m}-\frac{b_m^2}{2}\right)+\frac{1}{4}B_2b_m^2. \tag{4.8}$$

From (4.5) and (4.7), we get

$$c_m=-b_m. \tag{4.9}$$

Also from (4.6) ,(4.8) and (4.9), we obtain

$$a_{m+1}^2=\frac{B_1^3\left[c_{2m}+b_{2m}\right]}{2\left(m\left(3\lambda m^2+4\lambda m+2m^2+\lambda\right)B_1^2+2m^2(1+\lambda+\lambda m)^2(B_1-B_2)\right)},$$

which, in view of the inequalities $|c_{2m}|\leq 2$ and $|b_{2m}|\leq 2$ for $|a_{m+1}|$, we get the desired result as asserted in (4.1). Now, by a simple calculations from (4.5), (4.6), (4.8) and (4.9), we get

$$a_{2m+1}=\frac{B_1\left((m(2m+1)+\lambda m(m+1)(4m+1))c_{2m}+m(1+\lambda(m+1))b_{2m}\right)}{8m^3\left(1+4\lambda m+3\lambda+2\lambda^2(m+1)(2m+1)\right)}+\frac{(B_2-B_1)m(m+1)(1+\lambda+2\lambda m)c_m^2}{8m^3\left(1+4\lambda m+3\lambda+2\lambda^2(m+1)(2m+1)\right)},$$

which, in view of the inequalities $|c_m|\leq 2$, $|c_{2m}|\leq 2$ and $|b_{2m}|\leq 2$ for $|a_{2m+1}|$, we get the desired result as asserted in (4.2). □

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References

1. Ali, R.M., Lee, S.K., Ravichandran, V., Subramanian, S.: Coefficient estimates for bi-univalent Ma–Minda starlike and convex functions. Appl. Math. Lett. **25**(3), 344–351 (2012)

2. Brannan, B.A., Clunie, J.G.: Aspects of Contemporary Complex Analysis. Academic Press, London (1980)
3. Brannan, B.A., Taha, T.S.: On some classes of bi-univalent functions. Stud. Univ. Babes-Bolyai Math. **31**(2), 70–77 (1986)
4. Bulut, S.: Coefficient estimates for a class of analytic and bi-univalent functions. Novi Sad J. Math. **43**(2), 59–65 (2013)
5. Çağlar, M., Orhan, H., Yağmur, N.: Coefficient bounds for new subclasses of bi-univalent functions. Filomat **27**, 1165–1171 (2013)
6. Crişan, O.: Coefficient estimates for certain subclasses of bi-univalent functions. Gen. Math. Notes **16**, 93–102 (2013)
7. Deniz, E.: Certain subclasses of bi-univalent functions satisfying subordinate conditions. J. Class. Anal. **2**, 49–60 (2013)
8. Duren, P.L.: Univalent Functions. Springer, New York (1983)
9. Frasin, B.A., Aouf, M.K.: New subclasses of bi-univalent functions. Appl. Math. Lett. **24**, 1569–1573 (2011)
10. Goyal, S.P., Goswami, P.: Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives. J. Egypt. Math. Soc. **20**, 179–182 (2012)
11. Hayami, T., Owa, S.: Coefficient bounds for bi-univalent functions. Panam. Math. J. **22**(4), 15–26 (2012)
12. Lewin, M.: On a coefficient problem for bi-univalent functions. Proc. Am. Math. Soc. **18**, 63–68 (1967)
13. Ma, W.C., Minda, D.: A unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis, Tianjin, pp. 157–169. Internat. Press, Cambridge (1992)
14. Magesh, N., Rosy, T., Varma, S.: Coefficient estimate problem for a new subclass of bi-univalent functions. J. Complex Anal. **2013**, Article ID 474231, 1–3 (2013)
15. Magesh, N., Yamini, J.: Coefficient bounds for certainsubclasses of bi-univalent functions. Int. Math. Forum **8**, 1337–1344 (2013)
16. Murugusundaramoorthy, G., Magesh, N., Prameela. V.: Coefficient bounds for certain subclasses of bi-univalent functions. Abstr. Appl. Anal. **2013**, Article ID 573017, 1–3 (2013)
17. Pommerenke, C.H.: On the coefficients of close-to-convex functions. Mich. Math. J. **9**, 259–269 (1962)
18. Srivastava, H.M., Gaboury, S., Ghanim, F.: Coefficient estimates for some subclasses of $m-$fold symmetric bi-univalent functions, Acta Univ. Apulensis Math. Inform. **41**, 153–164 (2015)
19. Srivastava, H.M., Sivasubramanian, S., Sivakumar, R.: Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions. Tbil. Math. J. **7**(2), 1–10 (2014)
20. Srivastava, H.M., Mishra, A.K., Gochhayat, P.: Certain subclasses of analytic and bi-univalent functions. Appl. Math. Lett. **23**(10), 1188–1192 (2010)
21. Srivastava, H.M., Murugusundaramoorthy, G., Magesh, N.: Certain subclasses of bi-univalent functions associated with Hoholov operator. Glob. J. Math. Anal. **1**(2), 67–73 (2013)
22. Srivastava, H.M., Bulut, S., Çağlar, M., Yağmur, N.: Coefficient estimates for a general subclass of analytic and bi-univalent functions. Filomat **27**, 831–842 (2013)