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Results in Mathematics

Radii of α-Convexity of Some Normalized Bessel Functions of the First Kind

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Abstract. In this paper our aim is to determine the radii of $\alpha-$convexity of the normalized Bessel functions for two different kinds of normalization in the case when the order is between -2 and -1. The key tools in the proof of our main results are the Mittag-Leffler expansion for Bessel functions, properties of zeros of the Bessel functions and their derivatives and some inequalities for complex and real numbers.

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1. Introduction

For $r > 0$, $\mathcal{U}(r) = \{z \in \mathbb{C} : |z| < r\}$ denotes the disc of radius r centered at the origin. Let $f : \mathcal{U}(r_f) \to \mathbb{C}$ be the function defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

where r_f is the radius of convergence of the power series.

Let r be a real number with $r \in (0, r_f)$. We say that the function f defined by (1.1) is starlike in the disk $\mathcal{U}(r)$ if f is univalent in $\mathcal{U}(r)$, and $f(\mathcal{U}(r))$ is a starlike domain in $\mathbb{C}$ with respect to the origin. Analytically, the function f is starlike in $\mathcal{U}(r)$ if and only if $\mathrm{Re}(\frac{zf'(z)}{f(z)}) > 0$, $z \in \mathcal{U}(r)$. For $\beta \in [0, 1)$ we say

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that the function f is starlike of order β in $\mathcal{U}(r)$ if and only if $\mathrm{Re}(\frac{\mathrm{zf}'(\mathrm{z})}{\mathrm{f}(\mathrm{z})}) > \beta$, $z \in \mathcal{U}(r)$. We define by the real number

$$r_\beta^*(f) = \sup\left\{r \in (0, r_f) : \mathrm{Re}\left(\frac{\mathrm{zf}'(\mathrm{z})}{\mathrm{f}(\mathrm{z})}\right) > \beta \text{ for all } \mathrm{z} \in \mathcal{U}(\mathrm{r})\right\}$$

the radius of starlikeness of order β of the function f. Note that $r^*(f) = r_0^*(f)$ is the largest radius such that the image region $f(\mathcal{U}(r^*(f)))$ is a starlike domain with respect to the origin.

The function f defined by (1.1) is convex in the disk $\mathcal{U}(r)$ if f is univalent in $\mathcal{U}(r)$, and $f(\mathcal{U}(r))$ is a convex domain in $\mathbb{C}$. Analytically, the function f is convex in $\mathcal{U}(r)$ if and only if $\mathrm{Re}(1+\frac{\mathrm{zf}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})}) > 0$, $z \in \mathcal{U}(r)$. For $\beta \in [0, 1)$ we say that the function f is convex of order β in $\mathcal{U}(r)$ if and only if $\mathrm{Re}(1+\frac{\mathrm{zf}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})}) > \beta$, $z \in \mathcal{U}(r)$. The radius of convexity of order β of the function f is defined by the real number

$$r_\beta^c(f) = \sup\left\{r \in (0, r_f) : \mathrm{Re}\left(1 + \frac{\mathrm{zf}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})}\right) > \beta \text{ for all } \mathrm{z} \in \mathcal{U}(\mathrm{r})\right\}.$$

Note that $r^c(f) = r_0^c(f)$ is the largest radius such that the image region $f(\mathcal{U}(r^c(f)))$ is a convex domain.

Let α and β be two real numbers with $\alpha \in \mathbb{R}$ and $\beta \in [0, 1)$. We say that the function f is $\alpha-$ convex of order β in $\mathcal{U}(r)$ if and only if

$$\mathrm{Re}\left((1-\alpha)\frac{\mathrm{zf}'(\mathrm{z})}{\mathrm{f}(\mathrm{z})} + \alpha\left(1 + \frac{\mathrm{zf}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})}\right)\right) > \beta, \quad \mathrm{z} \in \mathcal{U}(\mathrm{r}).$$

The radius of $\alpha-$convexity of order β of the function f is defined by the real number

$$r_{\alpha,\beta}(f) = \sup\left\{r \in (0, r_f) : \mathrm{Re}\left((1-\alpha)\frac{\mathrm{zf}'(\mathrm{z})}{\mathrm{f}(\mathrm{z})} + \alpha\left(1 + \frac{\mathrm{zf}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})}\right)\right) > \beta \right.$$
$$\left. \text{for all } z \in \mathcal{U}(r)\right\}.$$

The radius of $\alpha-$convexity of order β is the generalization of the radius of starlikeness of order β and of the radius of convexity of order β. We have $r_{0,\beta}(f) = r_\beta^*(f)$ and $r_{1,\beta}(f) = r_\beta^c(f)$. For more details on starlike, convex and $\alpha-$convex functions we refer to [1–3] and to the references therein.

The Bessel function of the first kind of order v is defined by [4, p. 217]

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v+1)}\left(\frac{z}{2}\right)^{2n+v}, \quad z \in \mathbb{C}.$$

In this paper we focus on the following normalized forms

$$g_v(z) = 2^v\Gamma(v+1)z^{1-v}J_v(z) = z - \frac{1}{4(v+1)}z^3 + \frac{1}{32(v+1)(v+2)}z^5 - \cdots, \tag{1.2}$$

$$h_v(z) = 2^v \Gamma(v+1) z^{1-\frac{v}{2}} J_v(\sqrt{z}) = z - \frac{1}{4(v+1)} z^2 + \cdots , \tag{1.3}$$

where $-2 < v < -1$. Recently, Baricz et al. [5,6] and Baricz and Szász [7] obtained, respectively, the radius of starlikeness of order β, the radius of convexity of order β and the radius of $\alpha-$convexity of order β for the functions $g_v(z)$ and $h_v(z)$ in the case when $v > -1$. On the other hand, we know that Bessel functions have exactly two purely imaginary conjugate complex zeros and all the other zeros are real when $v \in (-2,-1)$ (see [8], p. 483). If $v \in (-2,-1)$, then to solve above radius problems the method which has been used in [5–7] is not applicable directly. In [9], Szász investigated the radius of starlikeness of order β for the functions $g_v(z)$ and $h_v(z)$ in the case when $v \in (-2,-1)$ by using some inequalities. Baricz and Szász [10] obtained the radius of convexity of order β for the functions $g_v(z)$ and $h_v(z)$ in the case when $v \in (-2,-1)$. In this paper, we deal with the radius of $\alpha-$convexity of order β for the functions $g_v(z)$ and $h_v(z)$ in the case when $(-2,-1)$.

In our study we need some results, which will be exposed in the next section.

2. Preliminaries

In order to prove the main results we need the following preliminary results. Here and in the sequel I_v denotes the modified Bessel function of the fist kind and order v. Note that $I_v(z) = i^{-v} J_v(iz)$ and $I_v(\sqrt{z}) = (-1)^{-\frac{v}{2}} J_v(\sqrt{-z})$.

Lemma 2.1 [8, p. 483, Hurwiz.]. *If $v \in (-2,-1)$, then $J_v(z) = 0$ has exactly two purely imaginary conjugate complex roots and all the other roots are real.*

The roots $z^{-v} J_v(z) = 0$ are taken to be $\pm j_{v,n}$, where $n \in \mathbb{N}^* = \{1,2,3,\ldots\}$. We may suppose without restricting the generality that $j_{v,1} = ia$, $a > 0$ and $0 < j_{v,2} < j_{v,3} < \cdots < j_{v,n} < \cdots$. It is well-known that $a < j_{v,2}$.

Lemma 2.2 [8, p. 502.]. *The following equality holds*

$$\sum_{n=1}^{\infty} \frac{1}{j_{v,n}^2} = \frac{1}{4(v+1)}. \tag{2.1}$$

Lemma 2.3 [5,7]. *Let $\alpha_{v,n}$ and $\beta_{v,n}$, $n \in \mathbb{N}$, respectively, denote the $n-$th roots of the equations $J_v(z) - zJ_{v+1}(z) = 0$ and $(2-v)J_v(z) + zJ_v'(z) = 0$, where $0 \le \mathrm{Re}\alpha_{\mathrm{v,1}} \le \mathrm{Re}\alpha_{\mathrm{v,2}} \le \cdots \le \mathrm{Re}\alpha_{\mathrm{v,n}} \le \cdots$ and $0 \le \mathrm{Re}\beta_{\mathrm{v,1}} \le \mathrm{Re}\beta_{\mathrm{v,2}} \le \cdots \le \mathrm{Re}\beta_{\mathrm{v,n}} \le \cdots$. The following developments hold for every $z \ne \alpha_{v,n}$, $z \ne \beta_{v,n}$*

$$\frac{zg_v'(z)}{g_v(z)} = 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{j_{v,n}^2 - z^2},\ 1 + \frac{zg_\nu''(z)}{g_\nu'(z)} = 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{\alpha_{v,n}^2 - z^2} \tag{2.2}$$

and

$$\frac{zh_v'(z)}{h_v(z)} = 1 - \sum_{n=1}^{\infty} \frac{z}{j_{v,n}^2 - z}, \quad 1 + \frac{zh_v''(z)}{h_v'(z)} = 1 - \sum_{n=1}^{\infty} \frac{z}{\beta_{v,n}^2 - z} \tag{2.3}$$

The series are uniformly convergent on every compact subset of $\mathbb{C}\backslash\{\pm j_{v,n} : n \in \mathbb{N}^*\}$.

There are some relations between $J_v(z)$ and the functions $g_v(z)$ and $h_v(z)$. For example (see [7]),

$$\frac{zg_v'(z)}{g_v(z)} = 1 - \frac{zJ_{v+1}(z)}{J_v(z)} \quad \text{and} \quad \frac{zg_v''(z)}{g_v'(z)} = z\frac{zJ_{v+2}(z) - 3J_{v+1}(z)}{J_v(z) - zJ_{v+1}(z)},$$

$$\frac{zh_v'(z)}{h_v(z)} = 1 - \frac{\sqrt{z}J_{v+1}(\sqrt{z})}{2J_v(\sqrt{z})} \text{ and } \frac{zh_v''(z)}{h_v'(z)} = \frac{zJ_{v+2}(\sqrt{z}) - 4\sqrt{z}J_{v+1}(\sqrt{z})}{4J_v(\sqrt{z}) - 2\sqrt{z}J_{v+1}(\sqrt{z})}.$$

Lemma 2.4 [5]. *If* $\upsilon \in \mathbb{C}$, $\delta \in \mathbb{R}$ *and* $\delta > |\upsilon|$, *then*

$$\frac{|\upsilon|}{\delta - |\upsilon|} \geq \operatorname{Re}\left(\frac{\upsilon}{\delta - \upsilon}\right) \geq \frac{-|\upsilon|}{\delta + |\upsilon|} \tag{2.4}$$

and

$$\operatorname{Re}\left(\frac{\upsilon}{\delta + \upsilon}\right) \geq \frac{-|\upsilon|}{\delta - |\upsilon|}. \tag{2.5}$$

Lemma 2.5 [9]. *If* $\upsilon \in \mathbb{C}$, $\gamma, \delta \in \mathbb{R}$ *and* $\gamma \geq \delta > r \geq |\upsilon|$, *then*

$$\operatorname{Re}\left(\frac{\upsilon^2}{(\delta + \upsilon)(\gamma - \upsilon)}\right) \leq \frac{\mathrm{r}^2}{(\delta - \mathrm{r})(\gamma + \mathrm{r})}. \tag{2.6}$$

Lemma 2.6. $j_{\nu,n}$ *denotes the* $n-1$*th positive zero of the Bessel function* J_ν *in case* $\nu \in (-2, -1)$. *The inequalities* $j_{\nu,n}^2 > a^2 > r \geq |z|$ *imply that*

$$\left|\frac{2a^2 j_{\nu,n}^2 + j_{\nu,n}^2 z - a^2 z}{(a^2 + z)^2 (j_{\nu,n}^2 - z)^2}\right| \leq \frac{2a^2 j_{\nu,n}^2 - j_{\nu,n}^2 r + a^2 r}{(a^2 - r)^2 (j_{\nu,n}^2 + r)^2}, \tag{2.7}$$

$$\left|\frac{z}{(a^2 + z)^2}\right| \leq \frac{r}{(a^2 - r)^2}. \tag{2.8}$$

The equalities hold if and only if $z = -r$.

Proof. According to the maximum principle it is enough to prove the inequality (2.7) in case of $z = re^{i\theta}$, that is

$$\left|\frac{2a^2 j_{\nu,n}^2 + j_{\nu,n}^2 re^{i\theta} - a^2 re^{i\theta}}{(a^2 + re^{i\theta})^2 (j_{\nu,n}^2 - re^{i\theta})^2}\right| \leq \frac{2a^2 j_{\nu,n}^2 - j_{\nu,n}^2 r + a^2 r}{(a^2 - r)^2 (j_{\nu,n}^2 + r)^2}. \tag{2.9}$$

Denoting $\alpha = \frac{j_{\nu,n}^2}{r}$ and $\beta = \frac{a^2}{r}$, the inequality (2.9) can be rewritten as follows

$$\left|\frac{2\alpha\beta + \alpha e^{i\theta} - \beta e^{i\theta}}{(\beta + e^{i\theta})^2 (\alpha - e^{i\theta})^2}\right| \leq \frac{2\alpha\beta - \alpha + \beta}{(\beta - 1)^2 (\alpha + 1)^2}, \tag{2.10}$$

where $\alpha > \beta > 1$. We will prove the inequality (2.10) in two steps.

First we show that

$$\left|\frac{1}{(\beta+e^{i\theta})(\alpha-e^{i\theta})}\right| \leq \frac{1}{(\beta-1)(\alpha+1)}, \quad \alpha>\beta>1 \tag{2.11}$$

If $t=\cos\theta$, then this inequality will be equivalent to

$$(\beta^2+2\beta t+1)(\alpha^2-2\alpha t+1) \geq (\beta-1)^2(\alpha+1)^2, \;\; \alpha>\beta>1, \; t\in[-1,1]. \tag{2.12}$$

In order to prove inequality (2.12) we define the function

$$u:[-1,1]\to\mathbb{R}, \quad u(t)=(\beta^2+2\beta t+1)(\alpha^2-2\alpha t+1).$$

Since $u''(t)=-8\alpha\beta<0, \; t\in[-1,1]$ it follows that u is a concave mapping, and consequently

$$u(t)\geq \min\{u(1),u(-1)\}=u(-1)=(\beta-1)^2(\alpha+1)^2.$$

Thus the proof of the inequality (2.11) is done. In the second step we will prove that:

$$\left|\frac{2\alpha\beta+\alpha e^{i\theta}-\beta e^{i\theta}}{(\beta+e^{i\theta})(\alpha-e^{i\theta})}\right| \leq \frac{2\alpha\beta-\alpha+\beta}{(\beta-1)(\alpha+1)}, \quad \theta\in[0,2\pi], \quad \alpha>\beta>1. \tag{2.13}$$

This inequality is equivalent to

$$\left|\frac{\alpha}{\alpha-e^{i\theta}}+\frac{\beta}{\beta+e^{i\theta}}\right| \leq \frac{\alpha}{\alpha+1}+\frac{\beta}{\beta-1}, \quad \theta\in[0,2\pi], \quad \alpha>\beta>1. \tag{2.14}$$

We have

$$\left|\frac{\alpha}{\alpha-e^{i\theta}}+\frac{\beta}{\beta+e^{i\theta}}\right| \leq \left|\frac{\alpha}{\alpha-e^{i\theta}}\right|+\left|\frac{\beta}{\beta-e^{i\theta}}\right|, \quad \theta\in[0,2\pi].$$

Consequently in order to prove (2.13) and (2.14) we have to show that

$$\left|\frac{\alpha}{\alpha-e^{i\theta}}\right|+\left|\frac{\beta}{\beta+e^{i\theta}}\right| \leq \frac{\alpha}{\alpha+1}+\frac{\beta}{\beta-1}, \quad \theta\in[0,2\pi], \quad \alpha>\beta>1. \tag{2.15}$$

Using again the notation $t=\cos\theta$ the inequality (2.16) can be rewritten as follows

$$\frac{\alpha}{(\alpha^2-2\alpha t+1)^{\frac{1}{2}}}+\frac{\beta}{(\beta^2+2\beta t+1)^{\frac{1}{2}}} \leq \frac{\alpha}{\alpha+1}+\frac{\beta}{\beta-1}, \;\; t\in[-1,1], \;\; \alpha>\beta>1. \tag{2.16}$$

We define the function

$$v:[-1,1]\to\mathbb{R}, \; v(t)=\frac{\alpha}{(\alpha^2-2\alpha t+1)^{\frac{1}{2}}}+\frac{\beta}{(\beta^2+2\beta t+1)^{\frac{1}{2}}}.$$

We have

$$v''(t)=3\alpha^3(\alpha^2-2\alpha t+1)^{-\frac{5}{2}}+3\beta^3(\beta^2-2\beta t+1)^{-\frac{5}{2}}>0, \; t\in[-1,1],$$

and this implies the convexity of v. Thus we get

$$v(t)\leq \max\{v(-1),v(1)\}=v(-1)=\frac{\alpha}{\alpha+1}+\frac{\beta}{\beta-1}, \quad t\in[-1,1],$$

which is equivalent to (2.16), and the proof of (2.13) is done.

Finally the inequalities (2.11) and (2.13) imply (2.10) and the proof is finished. Just like in the case of the first inequality we have to prove the inequality (2.8) in case $z = re^{i\theta}$. In this case the inequality (2.8) is equivalent to the obvious inequality $|a^2 + re^{i\theta}| \geq |a^2 - r|$, and the proof is done. □

3. Main Results

Theorem 3.1. *If* $\nu \in (-2, -1)$, $\alpha \geq 0$ *and* $\beta \in [0, 1)$, *then the radius of* $\alpha-$*convexity of order* β *of the function* g_ν *defined by* (1.2), *is* $r_{\alpha,\beta}(g_\nu) = r_0$, *where* r_0 *is the unique root of the equation*

$$1 + (1-\alpha)\frac{rI_{v+1}(r)}{I_v(r)} + \alpha r\frac{rI_{v+2}(r) + 3I_{v+1}(r)}{I_v(r) + rI_{v+1}(r)} = \beta,$$

in the interval $(0, r^*(g_\nu))$.

Proof. According to the equality (2.2) in Lemma 2.3, we have

$$\frac{zg_v'(z)}{g_v(z)} = 1 - 2\sum_{n=1}^{\infty}\frac{z^2}{j_{v,n}^2 - z^2}.$$

By using Lemma 2.1, the condition $\nu \in (-2, -1)$ implies $j_{\nu,1} = ia$, $a > 0$ and $j_{\nu,n} > 0$ for $n \in \{2, 3, \ldots\}$. Thus we get

$$\frac{zg_v'(z)}{g_v(z)} = 1 + \frac{2z^2}{a^2 + z^2} - 2\sum_{n=2}^{\infty}\frac{z^2}{j_{v,n}^2 - z^2}.$$

On the other hand, the equality (2.1) in Lemma 2.2 implies that

$$\frac{1}{a^2} = -\frac{1}{4(v+1)} + \sum_{n=2}^{\infty}\frac{1}{j_{v,n}^2}$$

and using this we obtain that

$$\frac{zg_\nu'(z)}{g_\nu(z)} = 1 - \frac{a^2}{2(1+\nu)}\frac{z^2}{a^2 + z^2} - 2\sum_{n=2}^{\infty}\frac{a^2 + j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4}{(a^2 + z^2)(j_{\nu,n}^2 - z^2)}. \quad (3.1)$$

On the other hand, taking $v = z^2$ in the inequality (2.5) and (2.6) we get the inequality

$$\mathrm{Re}\frac{\mathrm{z}^2}{\mathrm{a}^2 + \mathrm{z}^2} \geq \frac{-\mathrm{r}^2}{\mathrm{a}^2 - \mathrm{r}^2} \text{ and } \mathrm{Re}\frac{\mathrm{z}^4}{(\mathrm{a}^2 + \mathrm{z}^2)(\mathrm{j}_{\nu,\mathrm{n}}^2 - \mathrm{z}^2)} \leq \frac{\mathrm{r}^4}{(\mathrm{a}^2 - \mathrm{r}^2)(\mathrm{j}_{\nu,\mathrm{n}}^2 + \mathrm{r}^2)}$$

for all $|z| \leq r < a < j_{\nu,n}$. Taking into account that $-\frac{a^2}{2(1+\nu)} > 0$, these inequalities imply

$$\mathrm{Re}\left(\frac{\mathrm{z}\mathrm{g}_\nu'(\mathrm{z})}{\mathrm{g}_\nu(\mathrm{z})}\right) \geq 1 + \frac{a^2}{2(1+\nu)}\frac{r^2}{a^2 - r^2} - 2\sum_{n=2}^{\infty}\frac{a^2 + j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^4}{(a^2 - r^2)(j_{\nu,n}^2 + r^2)}$$

$$= \frac{irg_v'(ir)}{g_v(ir)} > 0, \;\; |z| < r < r^*(g_\nu). \tag{3.2}$$

The logarithmic differentiation of (3.1) gives

$$1+\frac{zg_\nu''(z)}{g_\nu'(z)} = \frac{zg_\nu'(z)}{g_\nu(z)} - \frac{\frac{a^4}{(1+\nu)}\frac{z^2}{(a^2+z^2)^2} + 2\sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4(4a^2j_{\nu,n}^2-2a^2z^2+2j_{\nu,n}^2z^2)}{(a^2+z^2)^2(j_{\nu,n}^2-z^2)^2}}{1-\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2+z^2} - 2\sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2}{(a^2+z^2)(j_{\nu,n}^2-z^2)}},$$

and we get

$$\begin{aligned}(1-\alpha)&\frac{zg_\nu'(z)}{g_\nu(z)} + \alpha\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)\\ &= \frac{zg_\nu'(z)}{g_\nu(z)} - \alpha\frac{\frac{a^4}{(1+\nu)}\frac{z^2}{(a^2+z^2)^2} + \sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4(2a^2j_{\nu,n}^2-a^2z^2+j_{\nu,n}^2z^2)}{(a^2+z)^2(j_{\nu,n}^2-z)^2}}{1-\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2+z^2} - 2\sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}}.\end{aligned} \tag{3.3}$$

The equality (3.3) implies

$$\begin{aligned}\mathrm{Re}&\left\{(1-\alpha)\frac{\mathrm{zg}_\nu'(\mathrm{z})}{\mathrm{g}_\nu(\mathrm{z})} + \alpha\left(1+\frac{\mathrm{zg}_\nu''(\mathrm{z})}{\mathrm{g}_\nu'(\mathrm{z})}\right)\right\}\\ &\geq \mathrm{Re}\left(\frac{\mathrm{zg}_\nu'(\mathrm{z})}{\mathrm{g}_\nu(\mathrm{z})}\right) - \alpha\left|\frac{\frac{\mathrm{a}^4}{(1+\nu)}\frac{\mathrm{z}^2}{(\mathrm{a}^2+\mathrm{z}^2)^2} + \sum_{\mathrm{n}=2}^\infty \frac{\mathrm{a}^2+\mathrm{j}_{\nu,\mathrm{n}}^2}{\mathrm{j}_{\nu,\mathrm{n}}^2}\frac{\mathrm{z}^4(2\mathrm{a}^2\mathrm{j}_{\nu,\mathrm{n}}^2-\mathrm{a}^2\mathrm{z}^2+\mathrm{j}_{\nu,\mathrm{n}}^2\mathrm{z}^2)}{(\mathrm{a}^2+\mathrm{z}^2)^2(\mathrm{j}_{\nu,\mathrm{n}}^2-\mathrm{z}^2)^2}}{1-\frac{\mathrm{a}^2}{2(1+\nu)}\frac{\mathrm{z}^2}{\mathrm{a}^2+\mathrm{z}^2} - 2\sum_{\mathrm{n}=2}^\infty \frac{\mathrm{a}^2+\mathrm{j}_{\nu,\mathrm{n}}^2}{\mathrm{j}_{\nu,\mathrm{n}}^2}\frac{\mathrm{z}^4}{(\mathrm{a}^2+\mathrm{z}^2)(\mathrm{j}_{\nu,\mathrm{n}}^2-\mathrm{z}^2)}}\right|.\end{aligned} \tag{3.4}$$

On the other hand, replacing z by z^2 in Lemma 2.6 the following inequality holds

$$\begin{aligned}&\left|\frac{\frac{a^4}{(1+\nu)}\frac{z^2}{(a^2+z^2)^2} + \sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4(2a^2j_{\nu,n}^2-a^2z^2+j_{\nu,n}^2z^2)}{(a^2+z^2)^2(j_{\nu,n}^2-z^2)^2}}{1-\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2+z^2} - 2\sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}}\right|\\ &\leq \frac{\left|\frac{a^4}{(1+\nu)}\frac{z^2}{(a^2+z^2)^2}\right| + \sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{2a^2j_{\nu,n}^2-a^2z^2+j_{\nu,n}^2z^2}{(a^2+z^2)^2(j_{\nu,n}^2-z^2)^2}\right||z^4|}{1-\left|\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2+z^2}\right| - 2\sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}\right|}\\ &= \frac{\frac{-a^4}{(1+\nu)}\left|\frac{z^2}{(a^2+z^2)^2}\right| + \sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{2a^2j_{\nu,n}^2-a^2z^2+j_{\nu,n}^2z^2}{(a^2+z^2)^2(j_{\nu,n}^2-z^2)^2}\right||z^4|}{1+\frac{a^2}{2(1+\nu)}\left|\frac{z^2}{a^2+z^2}\right| - 2\sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}\right|}\\ &\leq \frac{\frac{a^4}{(1+\nu)}\frac{-r^2}{(a^2-r^2)^2} + \sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{2a^2j_{\nu,n}^2+a^2r^2-j_{\nu,n}^2r^2}{(a^2-r^2)^2(j_{\nu,n}^2+r^2)^2}r^4}{1+\frac{a^2}{4(1+\nu)}\frac{r^2}{a^2-r^2} - 2\sum_{n=2}^\infty \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^4}{(a^2-r^2)(j_{\nu,n}^2+r^2)}}.\end{aligned} \tag{3.5}$$

The equality holds if and only if $z = ir$.

Finally the inequalities (3.2), (3.4) and (3.5) imply,

$$\begin{aligned}
&\mathrm{Re}\left\{(1-\alpha)\frac{\mathrm{z}\mathrm{g}_\nu'(\mathrm{z})}{\mathrm{g}_\nu(\mathrm{z})}+\alpha\left(1+\frac{\mathrm{z}\mathrm{g}_\nu''(\mathrm{z})}{\mathrm{g}_\nu'(\mathrm{z})}\right)\right\}\\
&\geq \frac{irg_r'(ir)}{g_r(ir)}-\alpha\frac{\frac{a^4}{(1+\nu)}\frac{-r^2}{(a^2-r^2)^2}+\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{2a^2j_{\nu,n}^2+a^2r^2-j_{\nu,n}^2r^2}{(a^2-r^2)^2(j_{\nu,n}^2+r^2)^2}r^4}{1+\frac{a^2}{4(1+\nu)}\frac{r^2}{a^2-r^2}-2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^4}{(a^2-r^2)(j_{\nu,n}^2+r^2)}}\\
&=(1-\alpha)\frac{irg_v'(ir)}{g_v(ir)}+\alpha\left(1+\frac{irg_v''(ir)}{g_v'(ir)}\right) \qquad (3.6)
\end{aligned}$$

where $|z|<r<r^*(g_\nu)$. From the inequality (3.6), we get

$$\inf_{|z|<r}\mathrm{Re}\left\{(1-\alpha)\frac{\mathrm{z}\mathrm{g}_\nu'(\mathrm{z})}{\mathrm{g}_\nu(\mathrm{z})}+\alpha\left(1+\frac{\mathrm{z}\mathrm{g}_\nu''(\mathrm{z})}{\mathrm{g}_\nu'(\mathrm{z})}\right)\right\}=(1-\alpha)\frac{\mathrm{irg}_\mathrm{v}'(\mathrm{ir})}{\mathrm{g}_\mathrm{v}(\mathrm{ir})}+\alpha\left(1+\frac{\mathrm{irg}_\mathrm{v}''(\mathrm{ir})}{\mathrm{g}_\mathrm{v}'(\mathrm{ir})}\right),$$

for every $r\in(0,r^*(g_\nu))$. Since

$$\phi:(0,r^*(g_\nu))\to\mathbb{R},\quad \phi(r)=(1-\alpha)\frac{irg_v'(ir)}{g_v(ir)}+\alpha\left(1+\frac{irg_v''(ir)}{g_v'(ir)}\right),$$

is a strictly decreasing continuous function and $\lim_{r\searrow 0}\phi(r)=1$, $\lim_{r\nearrow r^*(g_\nu)}\phi(r)=-\infty$ it follows that the equation $\phi(r)=\beta$ has an unique root r_0, and $r_{\alpha,\beta}(g_\nu)=r_0$. Therefore, by using the minimum principle of harmonic functions, the proof of Theorem 3.1 is completed. □

For convenience in the sequel we will use the following notation

$$F(\alpha,f(z))=(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right).$$

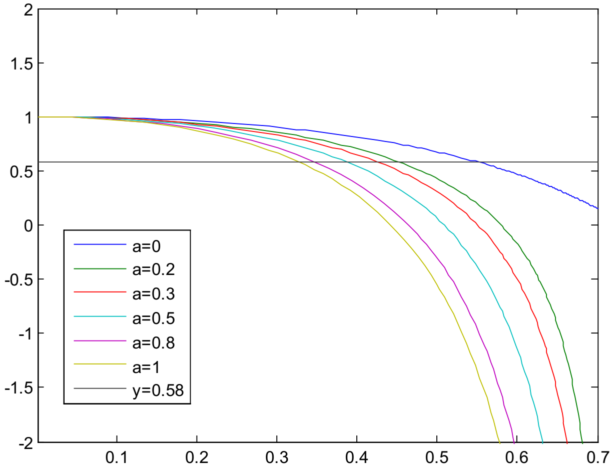

Figure 1. The graph of the function $r\longmapsto F(a,g_{-1.5}(ir))$ for $a\in\{0,0.2,0.3,0.5,0.8,1\}$ on $[0,0.7]$

Theorem 3.2. *If $\nu \in (-2,-1)$, $\alpha \geq 0$ and $\beta \in [0,1)$, then the radius of $\alpha-$convexity of order β of the function h_ν defined by (1.3), is $r_{\alpha,\beta}(h_\nu) = r_1$, where r_1 is the unique root of the equation*

$$1+(1-\alpha)\frac{\sqrt{r}I_{v+1}(\sqrt{r})}{2I_v(\sqrt{r})}+\alpha\frac{rI_{v+2}(\sqrt{r})+4\sqrt{r}I_{v+1}(\sqrt{r})}{4I_v(\sqrt{r})+2\sqrt{r}I_{v+1}(\sqrt{r})}=\beta,$$

in the interval $(0, r^*(h_\nu))$.

Proof. Similarly to the proof of Theorem 3.1, we deduce from Lemma 2.2 and the equality (2.3) in Lemma 2.3 the equality

$$z\frac{h_\nu'(z)}{h_\nu(z)}=1-\frac{a^2}{4(1+\nu)}\frac{z}{a^2+z}-\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2}{(a^2+z)(j_{\nu,n}^2-z)}. \tag{3.7}$$

On the other hand, taking $v = z$ in the inequality (2.5) and (2.6) we get the inequality

$$\mathrm{Re}\frac{\mathrm{z}}{\mathrm{a}^2+\mathrm{z}}\geq\frac{-\mathrm{r}}{\mathrm{a}^2-\mathrm{r}} \text{ and } \mathrm{Re}\frac{\mathrm{z}^2}{(\mathrm{a}^2+\mathrm{z})(\mathrm{j}_{\nu,\mathrm{n}}^2-\mathrm{z})}\leq\frac{\mathrm{r}^2}{(\mathrm{a}^2-\mathrm{r})(\mathrm{j}_{\nu,\mathrm{n}}^2+\mathrm{r})}$$

for all $|z| \leq r < a^2 < j_{\nu,n}^2$ and $n \in \{2,3,\ldots\}$. Taking into account that $-\frac{a^2}{4(1+\nu)} > 0$, these inequalities imply

$$\begin{aligned}\mathrm{Re}\left(\mathrm{z}\frac{\mathrm{h}_\nu'(\mathrm{z})}{\mathrm{h}_\nu(\mathrm{z})}\right)&\geq 1+\frac{a^2}{4(1+\nu)}\frac{r}{a^2-r}-\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^2}{(a^2-r)(j_{\nu,n}^2+r)}\\&=\frac{-rh_\nu'(-r)}{h_\nu(-r)}>0,\ |z|<r<r^*(h_\nu).\end{aligned} \tag{3.8}$$

The logarithmic differentiation of (3.7) gives

$$1+z\frac{h_\nu''(z)}{h_\nu'(z)}=z\frac{h_\nu'(z)}{h_\nu(z)}-\frac{\frac{a^4}{4(1+\nu)}\frac{z}{(a^2+z)^2}+\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2(2a^2j_{\nu,n}^2-a^2z+j_{\nu,n}^2z)}{(a^2+z)^2(j_{\nu,n}^2-z)^2}}{1-\frac{a^2}{4(1+\nu)}\frac{z}{a^2+z}-\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2}{(a^2+z)(j_{\nu,n}^2-z)}},$$

and we get

$$\begin{aligned}(1-\alpha)\frac{zh_\nu'(z)}{h_\nu(z)}+\alpha\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)&=z\frac{h_\nu'(z)}{h_\nu(z)}\\&-\alpha\frac{\frac{a^4}{4(1+\nu)}\frac{z}{(a^2+z)^2}+\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2(2a^2j_{\nu,n}^2-a^2z+j_{\nu,n}^2z)}{(a^2+z)^2(j_{\nu,n}^2-z)^2}}{1-\frac{a^2}{4(1+\nu)}\frac{z}{a^2+z}-\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2}{(a^2+z)(j_{\nu,n}^2-z)}}.\end{aligned} \tag{3.9}$$

The equality (3.9) implies

$$\mathrm{Re}\left\{(1-\alpha)\frac{\mathrm{z}\mathrm{h}_\nu'(\mathrm{z})}{\mathrm{h}_\nu(\mathrm{z})}+\alpha\left(1+\frac{\mathrm{z}\mathrm{h}_\nu''(\mathrm{z})}{\mathrm{h}_\nu'(\mathrm{z})}\right)\right\}$$
$$\geq \mathrm{Re}\left(\frac{\mathrm{z}\mathrm{h}_\nu'(\mathrm{z})}{\mathrm{h}_\nu(\mathrm{z})}\right)-\alpha\left|\frac{\frac{\mathrm{a}^4}{4(1+\nu)}\frac{\mathrm{z}}{(\mathrm{a}^2+\mathrm{z})^2}+\sum_{\mathrm{n}=2}^{\infty}\frac{\mathrm{a}^2+\mathrm{j}_{\nu,\mathrm{n}}^2}{\mathrm{j}_{\nu,\mathrm{n}}^2}\frac{\mathrm{z}^2(2\mathrm{a}^2\mathrm{j}_{\nu,\mathrm{n}}^2-\mathrm{a}^2\mathrm{z}+\mathrm{j}_{\nu,\mathrm{n}}^2\mathrm{z})}{(\mathrm{a}^2+\mathrm{z})^2(\mathrm{j}_{\nu,\mathrm{n}}^2-\mathrm{z})^2}}{1-\frac{\mathrm{a}^2}{4(1+\nu)}\frac{\mathrm{z}}{\mathrm{a}^2+\mathrm{z}}-\sum_{\mathrm{n}=2}^{\infty}\frac{\mathrm{a}^2+\mathrm{j}_{\nu,\mathrm{n}}^2}{\mathrm{j}_{\nu,\mathrm{n}}^2}\frac{\mathrm{z}^2}{(\mathrm{a}^2+\mathrm{z})(\mathrm{j}_{\nu,\mathrm{n}}^2-\mathrm{z})}}\right|. \tag{3.10}$$

On the other hand, using Lemma 2.6 we infer

$$\left|\frac{\frac{a^4}{4(1+\nu)}\frac{z}{(a^2+z)^2}+\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2(2a^2j_{\nu,n}^2-a^2z+j_{\nu,n}^2z)}{(a^2+z)^2(j_{\nu,n}^2-z)^2}}{1-\frac{a^2}{4(1+\nu)}\frac{z}{a^2+z}-\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2}{(a^2+z)(j_{\nu,n}^2-z)}}\right|$$
$$\leq \frac{\left|\frac{a^4}{4(1+\nu)}\frac{z}{(a^2+z)^2}\right|+\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{2a^2j_{\nu,n}^2-a^2z+j_{\nu,n}^2z}{(a^2+z)^2(j_{\nu,n}^2-z)^2}\right||z^2|}{\left|1-\frac{a^2}{4(1+\nu)}\frac{z}{a^2+z}-\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^2}{(a^2+z)(j_{\nu,n}^2-z)}\right|}$$
$$\leq \frac{\left|\frac{a^4}{4(1+\nu)}\frac{z}{(a^2+z)^2}\right|+\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{2a^2j_{\nu,n}^2-a^2z+j_{\nu,n}^2z}{(a^2+z)^2(j_{\nu,n}^2-z)^2}\right||z^2|}{\mathrm{Re}\left(1-\frac{\mathrm{a}^2}{4(1+\nu)}\frac{\mathrm{z}}{\mathrm{a}^2+\mathrm{z}}-\sum_{\mathrm{n}=2}^{\infty}\frac{\mathrm{a}^2+\mathrm{j}_{\nu,\mathrm{n}}^2}{\mathrm{j}_{\nu,\mathrm{n}}^2}\frac{\mathrm{z}^2}{(\mathrm{a}^2+\mathrm{z})(\mathrm{j}_{\nu,\mathrm{n}}^2-\mathrm{z})}\right)}$$
$$\leq \frac{\frac{a^4}{4(1+\nu)}\frac{-r}{(a^2-r)^2}+\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{2a^2j_{\nu,n}^2+a^2r-j_{\nu,n}^2r}{(a^2-r)^2(j_{\nu,n}^2+r)^2}r^2}{1+\frac{a^2}{4(1+\nu)}\frac{r}{a^2-r}-\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^2}{(a^2-r)(j_{\nu,n}^2+r)}}. \tag{3.11}$$

The equality holds if and only if $z=-r$. Finally the inequalities (3.8), (3.10) and (3.11) imply

$$\mathrm{Re}\left\{(1-\alpha)\frac{\mathrm{z}\mathrm{h}_\nu'(\mathrm{z})}{\mathrm{h}_\nu(\mathrm{z})}+\alpha\left(1+\frac{\mathrm{z}\mathrm{h}_\nu''(\mathrm{z})}{\mathrm{h}_\nu'(\mathrm{z})}\right)\right\}\geq\frac{-\mathrm{r}\mathrm{h}_\nu'(-\mathrm{r})}{\mathrm{h}_\nu(-\mathrm{r})}$$
$$-\alpha\frac{\frac{a^4}{4(1+\nu)}\frac{-r}{(a^2-r)^2}+\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^2(2a^2j_{\nu,n}^2+a^2r-j_{\nu,n}^2r)}{(a^2-r)^2(j_{\nu,n}^2+r)^2}}{1+\frac{a^2}{4(1+\nu)}\frac{r}{a^2-r}-\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^2}{(a^2-r)(j_{\nu,n}^2+r)}}$$
$$=(1-\alpha)\frac{-rh_\nu'(-r)}{h_\nu(-r)}+\alpha\left(1+\frac{-rh_\nu''(-r)}{h_\nu'(-r)}\right), \tag{3.12}$$

where $|z| < r < r^*(h_\nu)$. The inequality (3.12) implies that

$$\inf_{|z|<r} \mathrm{Re}\left\{(1-\alpha)\frac{\mathrm{z}\mathrm{h}_\nu'(\mathrm{z})}{\mathrm{h}_\nu(\mathrm{z})} + \alpha\left(1+\frac{\mathrm{z}\mathrm{h}_\nu''(\mathrm{z})}{\mathrm{h}_\nu'(\mathrm{z})}\right)\right\}$$
$$= (1-\alpha)\frac{-rh_\nu'(-r)}{h_\nu(-r)} + \alpha\left(1+\frac{-rh_\nu''(-r)}{h_\nu'(-r)}\right),$$

for every $r \in (0, r^*(h_\nu))$. Since

$$\tau : (0, r^*(h_\nu)) \to \mathbb{R}, \quad \tau(r) = (1-\alpha)\frac{-rh_\nu'(-r)}{h_\nu(-r)} + \alpha\left(1+\frac{-rh_\nu''(-r)}{h_\nu'(-r)}\right),$$

is a strictly decreasing continuous function and $\lim_{r\searrow 0}\tau(r) = 1$, $\lim_{r\nearrow r^*(h_\nu)} \tau(r) = -\infty$ it follows that the equation $\tau(r) = \beta$ has an unique root r_1, and $r_{\alpha,\beta}(h_\nu) = r_1$. Finally, by using the minimum principle of harmonic functions, the proof of Theorem 3.2 is completed. □

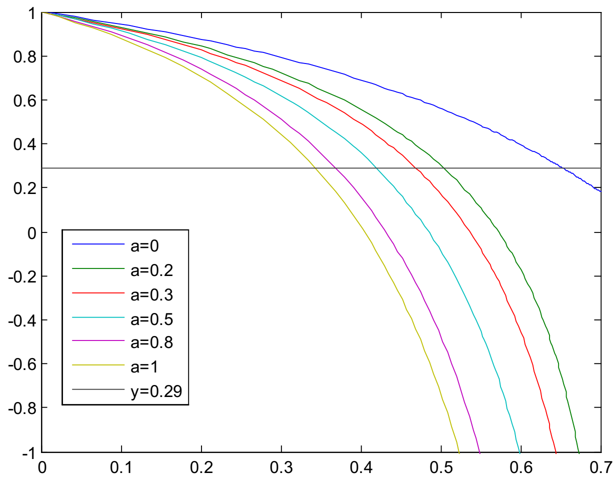

FIGURE 2. The graph of the function $r \longmapsto F(a, h_{-1.5}(-r))$ for $a \in \{0, 0.2, 0.3, 0.5, 0.8, 1\}$ on $[0, 0.7]$

If we take $\alpha = 0$ and $\alpha = 1$ in the Theorems 3.1, and 3.2 ,we obtain the results of Szász in [9], and Baricz and Szász [10], respectively. Our results is a common generalization of these results. Figures 1 and 2 illustrates the fact that for $\alpha \in [0, 1]$ the radius $\alpha-$convexity of the functions g_ν and h_ν is between its radii of convexity and starlikeness, that is, $r^c_\beta(g_\nu) < r_{\alpha,\beta}(g_\nu) < r^*_\beta(g_\nu)$ and $r^c_\beta(h_\nu) < r_{\alpha,\beta}(h_\nu) < r^*_\beta(h_\nu)$ for all $\alpha \in (0, 1)$, $\beta \in [0, 1)$ and $\nu \in (-2, -1)$. For $v = -1.5$ and $\alpha \in \{0, 0.2, 0.3, 0.5, 0.8, 1\}$ we considered the particular cases when $\beta = 0.58$, and $\beta = 0.29$, respectively, in Figs. 1 and 2.

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